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The Baxter–Bazhanov–Stroganov model: separation of variables and the Baxter equation

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Abstract
The Baxter–Bazhanov–Stroganov model (also known as the $\tau^{(2)}$ model) has attracted much interest because it provides a tool for solving the integrable chiral $\mathbb{Z}_N$-Potts model. It can be formulated as a face spin model or via cyclic L-operators. Using the latter formulation and the Sklyanin–Kharchev–Lebedev approach, we give the explicit derivation of the eigenvectors of the component $B_n(\lambda)$ of the monodromy matrix for the fully inhomogeneous chain of finite length. For the periodic chain, we obtain the Baxter T-Q-equations via separation of variables. The functional relations for the transfer matrices of the $\tau^{(2)}$ model guarantee nontrivial solutions to the Baxter equations. For the $N = 2$ case, which is the free fermion point of a generalized Ising model, the Baxter equations are solved explicitly.

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1. Introduction

The aim of this paper is the explicit construction of eigenvectors of the transfer matrix for the finite-size inhomogeneous periodic Baxter–Bazhanov–Stroganov model (BBS model) also known as the $\tau^{(2)}$ model [1–4]. This is an N-state spin lattice model, intimately related to the integrable chiral Potts model. The connection between the six-vertex model, the BBS model and the chiral Potts model gives the possibility of formulating a system of functional relations [3, 4] for the transfer matrices of these models. Solving these systems is the basic method for calculating the eigenvalues of the transfer matrix of the chiral Potts model [5], and under some analyticity assumptions, to derive the free energy of this model [5] and its order parameter [6].

In general, for the BBS model there is no Bethe pseudovacuum state and so the algebraic Bethe ansatz cannot be used. Therefore, in order to achieve our goal, we shall use the formulation of the BBS model in terms of cyclic L-operators first introduced by Korepanov [7] and Bazhanov–Stroganov [3] and adapt the Sklyanin–Kharchev–Lebedev method of separation of variables (SoV) [8–11] for solving the BBS eigenvector problem. The fusion equations will provide the existence of solutions to the Baxter equations.

The paper is organized as follows. After defining the BBS model as a statistical face spin model, we give the vertex formulation of the model in terms of a cyclic L-operator and conclude the introduction explaining the two basic steps involved in the SoV method. Section 2 deals with the solution of the auxiliary problem, leaving for section 3 the lengthy inductive proof of the main formula. Section 4 derives the action of the diagonal component D of the L-operator on the auxiliary eigenstates. Then in section 5 we come to the periodic model and in deriving the Baxter T-Q-equations we show the role of the fusion equations for solving these Baxter equations. In section 6, we apply these results to the homogeneous $N = 2$ case and calculate the eigenvalues and eigenvectors of the homogeneous $N = 2$ BBS model which is the free fermion point of a generalized Ising model. Section 7 gives our conclusions. In the appendix, we show a strong simplification occurring if the BBS model is homogeneous.

1.1. The BBS model

Following the notation of a recent paper of Baxter [1], we define the BBS model as a statistical model of short-range interacting spins placed at the vertices of a rectangular lattice. We label the spin variables $\mathrm{s}_{x,y}$ by a pair (x, y) of integers: $x = 1, \dots, n$ and $y = 1, \dots, m$. Each spin variable $\mathrm{s}_{x,y}$ takes N values ($N \geqslant 2$): $0, 1, \dots, N-1$. The model shall have $\mathbb{Z}_N$-symmetry and we may extend the range of the spins $\mathrm{s}_{x,y}$ to all integers identifying two values if their difference is a multiple of N. The model has a chiral restriction on the values of vertically neighbouring spins:

$$\mathrm{s}_{x,y} - \mathrm{s}_{x,y+1} = 0 \text{ or } 1 \quad \mathrm{mod}\ N. \tag{1}$$

In the following, we will consider the spin variables on two adjacent rows: (k, l) and $(k, l+1)$, where l is fixed and $k = 1, \dots, n$. Let us denote $\mathrm{s}_{k,l} = \gamma_k$ and $\mathrm{s}_{k,l+1} = \gamma'_k$. The model depends on the parameters t_q and $a'_k, b'_k, c'_k, d'_k, a''_k, b''_k, c''_k, d''_k$, $k = 1, 2, \dots, n$. Each square plaquette of the lattice has the Boltzmann weight (see figure 1)

$$W_\tau(\gamma_{k-1}, \gamma_k; \gamma'_{k-1}, \gamma'_k) = \sum_{m_{k-1}=0}^{1} \omega^{m_{k-1}(\gamma'_k - \gamma_{k-1})} (-\omega t_q)^{\gamma_k - \gamma'_k - m_{k-1}}$$
$$\times F'_{k-1}(\gamma_{k-1} - \gamma'_{k-1}, m_{k-1}) F''_k(\gamma_k - \gamma'_k, m_{k-1}), \tag{2}$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/N}$ and

$$F'_k(0,0) = 1, \qquad F'_k(0,1) = -\omega t_q \frac{c'_k}{b'_k}, \qquad F'_k(1,0) = \frac{d'_k}{b'_k}, \qquad F'_k(1,1) = -\omega \frac{a'_k}{b'_k},$$

and expressions for $F''_k(\gamma_k - \gamma'_k, m_{k-1})$ are obtained from $F'_k(\gamma_k - \gamma'_k, m_k)$ by substitutions $a'_k, b'_k, c'_k, d'_k \to a''_k, b''_k, c''_k, d''_k$.

We will consider the periodic boundary condition $\gamma_{n+1} = \gamma_1, \gamma'_{n+1} = \gamma'_1$, where n is the number of sites on the lattice along the horizontal axis. The transfer matrix of the periodic BBS model is an $N^n \times N^n$ matrix with matrix elements

$$\mathbf{t}_n(\gamma, \gamma') = \prod_{k=2}^{n+1} W_\tau(\gamma_{k-1}, \gamma_k; \gamma'_{k-1}, \gamma'_k) \tag{3}$$

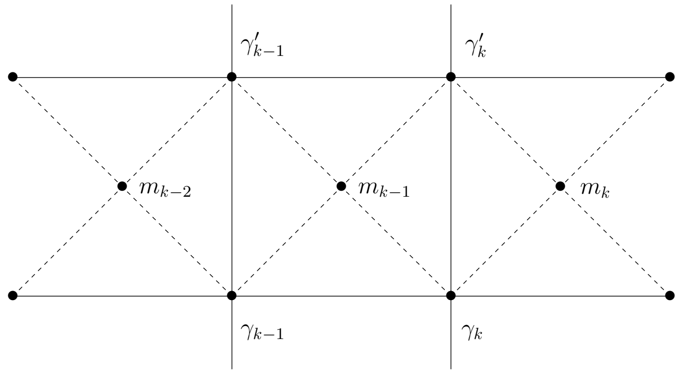

Figure 1. The triangle with vertices marked by the spin variables $\gamma_{k-1}, \gamma'_{k-1}, m_{k-1}$ corresponds to the function $F'_{k-1}(\gamma_{k-1} - \gamma'_{k-1}, m_{k-1})$ in (2); the triangle $\gamma_k, \gamma'_k, m_{k-1}$ to $F''_k(\gamma_k - s'_k, m_{k-1})$.

labelled by the sets of spin variables $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ and $\gamma' = \{\gamma'_1, \gamma'_2, \ldots, \gamma'_n\}$ of two neighbour rows.

Considering $m_k, k = 1, \ldots, n$, in (2) as auxiliary spin variables which take the two values 0 and 1, we can rewrite the transfer matrix (3) in a vertex formulation associating a statistical weight not with the plaquettes but with vertices, each of them relating four spins: $m_{k-1}, m_k, \gamma_k, \gamma'_k$ (see figure 1). Then the weight associated with the kth vertex is

$$\ell_k(t_q; m_{k-1}, m_k; \gamma_k, \gamma'_k) = \omega^{m_{k-1}\gamma'_k - m_k\gamma_k}(-\omega t_q)^{\gamma_k - \gamma'_k - m_{k-1}} F''_k(\gamma_k - \gamma'_k, m_{k-1}) F'_k(\gamma_k - \gamma'_k, m_k) \tag{4}$$

and the transfer matrix (3) can be rewritten as

$$\mathbf{t}_n(\gamma, \gamma') = \sum_{m_1,\ldots,m_n} \prod_{k=2}^{n+1} \ell_k(t_q; m_{k-1}, m_k; \gamma_k, \gamma'_k). \tag{5}$$

1.2. The L-operator formulation of the BBS model

For our construction of the BBS model eigenvectors, we will use a description of this model as a quantum chain model as introduced in [3, 7]. With each site k of the quantum chain we associate the cyclic L-operator acting in a two-dimensional auxiliary space

$$L_k(\lambda) = \begin{pmatrix} 1 + \lambda\varkappa_k \mathbf{v}_k & \lambda \mathbf{u}_k^{-1}(a_k - b_k\mathbf{v}_k) \\ \mathbf{u}_k(c_k - d_k\mathbf{v}_k) & \lambda a_k c_k + \mathbf{v}_k b_k d_k/\varkappa_k \end{pmatrix}, \qquad k = 1, 2, \ldots, n. \tag{6}$$

At each site k, we define ultra-local Weyl elements $\mathbf{u}_k$ and $\mathbf{v}_k$ obeying the commutation rules and normalization

$$\begin{aligned} &\mathbf{u}_j\mathbf{u}_k = \mathbf{u}_k\mathbf{u}_j, \qquad \mathbf{v}_j\mathbf{v}_k = \mathbf{v}_k\mathbf{v}_j, \qquad \mathbf{u}_j\mathbf{v}_k = \omega^{\delta_{j,k}}\mathbf{v}_k\mathbf{u}_j, \\ &\omega = \mathrm{e}^{2\pi \mathrm{i}/N}, \qquad \mathbf{u}_k^N = \mathbf{v}_k^N = 1. \end{aligned} \tag{7}$$

In (6), λ is the spectral parameter and we have five parameters $\varkappa_k, a_k, b_k, c_k, d_k$ per site. At each site k, we define an N-dimensional linear space (quantum space) $\mathcal{V}_k$ with the basis $|\gamma\rangle_k, \gamma \in \mathbb{Z}_N$, and natural scalar product ${}_k\langle\gamma'|\gamma\rangle_k = \delta_{\gamma',\gamma}$. In $\mathcal{V}_k$, the Weyl elements $\mathbf{u}_k$ and $\mathbf{v}_k$ act by the formulae

$$\mathbf{u}_k|\gamma\rangle_k = \omega^\gamma|\gamma\rangle_k, \qquad \mathbf{v}_k|\gamma\rangle_k = |\gamma + 1\rangle_k. \tag{8}$$

The correspondence between the lattice BBS model and its quantum chain analogue is established through the relation

$$\ell_k(t_q; m_{k-1}, m_k; \gamma_k, \gamma_k') = {}_k\langle\gamma|L_k(\lambda)_{m_{k-1},m_k}|\gamma'\rangle_k \tag{9}$$

and the following connection between the parameters of these models

$$\lambda = -\omega t_q, \qquad \varkappa_k = \frac{d_k'}{b_k'}\frac{d_k''}{b_k''}, \qquad a_k = \frac{c_k''}{b_k''},$$
$$b_k = \omega\frac{a_k''}{b_k''}\frac{d_k'}{b_k'}, \qquad c_k = \frac{c_k''}{b_k''}, \qquad d_k = \frac{a_k''}{b_k''}\frac{d_k'}{b_k'}.$$

We extend the action of the operators $\mathbf{u}_k, \mathbf{v}_k$ to $\mathcal{V}^{(n)} = \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_n$ defining this action to be trivial in all $\mathcal{V}_s$ with $s \neq k$. The monodromy matrix for the quantum chain with n sites is defined as

$$T_n(\lambda) = L_1(\lambda)L_2(\lambda)\cdots L_n(\lambda) = \begin{pmatrix} A_n(\lambda) & B_n(\lambda) \\ C_n(\lambda) & D_n(\lambda) \end{pmatrix}. \tag{10}$$

The transfer matrix (5) is obtained taking the trace in the auxiliary space

$$\mathbf{t}_n(\lambda) = \operatorname{tr} T_n(\lambda) = A_n(\lambda) + D_n(\lambda). \tag{11}$$

This quantum chain is integrable because the L-operators (6) are intertwined by the twisted six-vertex R-matrix at root of unity

$$R(\lambda, \nu) = \begin{pmatrix} \lambda - \omega\nu & 0 & 0 & 0 \\ 0 & \omega(\lambda - \nu) & \lambda(1-\omega) & 0 \\ 0 & \nu(1-\omega) & \lambda - \nu & 0 \\ 0 & 0 & 0 & \lambda - \omega\nu \end{pmatrix}, \tag{12}$$

$$R(\lambda, \nu)L_k^{(1)}(\lambda)L_k^{(2)}(\nu) = L_k^{(2)}(\nu)L_k^{(1)}(\lambda)R(\lambda, \nu), \tag{13}$$

where $L_k^{(1)}(\lambda) = L_k(\lambda) \otimes \mathbb{I}$, $L_k^{(2)}(\lambda) = \mathbb{I} \otimes L_k(\lambda)$. Relation (13) leads to $[\mathbf{t}_n(\lambda), \mathbf{t}_n(\mu)] = 0$ and so $\mathbf{t}_n(\lambda)$ is the generating function for the commuting set of non-local and non-Hermitian Hamiltonians $\mathbf{H}_0, \ldots, \mathbf{H}_n$ of the model:

$$\mathbf{t}_n(\lambda) = \mathbf{H}_0 + \mathbf{H}_1\lambda + \cdots + \mathbf{H}_{n-1}\lambda^{n-1} + \mathbf{H}_n\lambda^n. \tag{14}$$

The lowest and highest Hamiltonians can easily be written explicitly in terms of the global $\mathbb{Z}_N$-charge rotation operator $\mathbf{V}_n$:

$$\mathbf{H}_0 = 1 + \mathbf{V}_n\prod_{k=1}^{n}\frac{b_k d_k}{\varkappa_k}, \qquad \mathbf{H}_n = \prod_{k=1}^{n} a_k c_k + \mathbf{V}_n\prod_{k=1}^{n}\varkappa_k, \qquad \mathbf{V}_n = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_n. \tag{15}$$

It also follows from the intertwining relation (13) that $B_n(\lambda)$ is the generating function for another commuting set of operators $\mathbf{h}_1, \ldots, \mathbf{h}_n$:

$$[B_n(\lambda), B_n(\mu)] = 0, \qquad B_n(\lambda) = \mathbf{h}_1\lambda + \mathbf{h}_2\lambda^2 + \cdots + \mathbf{h}_n\lambda^n. \tag{16}$$

The great interest in the BBS chain model is due to its relation to the integrable chiral Potts model. In [3, 7], it was observed that besides the intertwining relations (13), the L-operators (6) satisfy a second intertwining relation in the Weyl quantum space:

$$\sum_{\beta_1\beta_2,j} \mathsf{S}_{\alpha_1\alpha_2;\beta_1\beta_2}(p, p', q, q')L_{i_1 j}^{\beta_1\gamma_1}(\lambda; p, p')L_{j i_2}^{\beta_2,\gamma_2}(\lambda; q, q')$$
$$= \sum_{\beta_1\beta_2,j} L_{i_1 j}^{\alpha_2\beta_2}(\lambda; q, q')L_{j i_2}^{\alpha_1\beta_1}(\lambda; p, p')\mathsf{S}_{\beta_1,\beta_2;\gamma_1,\gamma_2}(p, p', q, q'), \tag{17}$$

if the parameters are chosen as

$$\varkappa_k = \frac{y_{q_k} y_{q'_k}}{\mu_{q_k}\mu_{q'_k}}, \qquad a_k = x_{q_k}, \qquad b_k = \frac{y_{q'_k}}{\mu_{q_k}\mu_{q'_k}}, \qquad c_k = \omega x_{q'_k}, \qquad d_k = \frac{y_{q_k}}{\mu_{q_k}\mu_{q'_k}}, \tag{18}$$

where x_{q_k}, y_{q_k}, μ_{q_k} (analogously for $x_{q'_k}$, etc) satisfy the chiral Potts model constraints

$$x_{q_k}^N + y_{q_k}^N = \mathsf{k}\left(x_{q_k}^N y_{q_k}^N + 1\right), \qquad \mathsf{k} x_{q_k}^N = 1 - \mathsf{k}'\mu_{q_k}^{-N}, \qquad \mathsf{k} y_{q_k}^N = 1 - \mathsf{k}'\mu_{q_k}^N, \qquad \mathsf{k}^2 + \mathsf{k}'^2 = 1. \tag{19}$$

Here k and k' are temperature parameters. In (17), we have written the spin matrix elements of the L-operators with the parametrization (18) as $L_{i,j}^{\alpha,\beta}(\lambda, q, q')$, where $i, j = 0, 1$ are the components in the auxiliary space and Greek indices $\alpha, \beta = 0, \ldots, N-1$ denote the components in the quantum space (8), suppressing the site index k. The matrix S turns out to be the product of four chiral Potts–Boltzmann weights [3]

$$\mathsf{S}_{\alpha_1\alpha_2,\beta_1\beta_2}(p, p', q, q') = W_{pq'}(\alpha_1 - \alpha_2) W_{p'q}(\beta_2 - \beta_1) \bar{W}_{pq}(\beta_2 - \alpha_1) \bar{W}_{p'q'}(\beta_1 - \alpha_2). \tag{20}$$

In the parametrization (18) of the BBS model, there are various functional relations to the chiral Potts model transfer matrix which have been used to obtain explicit solutions for the chiral Potts eigenvalues [4]. Only further restricting the parameters to the 'superintegrable chiral Potts' case:

$$a_k = \omega^{-1} b_k = c_k = d_k = \varkappa_k = 1 \tag{21}$$

allows us to solve the BBS model by algebraic Bethe ansatz, see e.g. [12, 13, 15]. In this form, Baxter in 1989 first obtained the BBS model as an 'inverse' of the superintegrable chiral Potts model, see equations (8.13), (8.14) of [2]. In this paper, we shall follow [1] in not using restrictions like (19) on the parameters $\varkappa_k, a_k, b_k, c_k, d_k$. We only shall exclude the superintegrable case (21).

It was shown in the paper [16] that the $N = 2$ BBS model is equivalent to the generalized Ising model at the free fermion point. The results of our paper permit us to obtain the transfer-matrix eigenvectors for this model. Recently, the interesting paper [17] appeared, where these eigenvectors were constructed using the Grassmann functional integral.

1.3. Functional Bethe ansatz and SoV

The construction of common eigenvectors of the set commuting integrals (14) will be solved in two main steps, which generally can be formulated as follows.

First, for the given quantum integrable chain type model one has to find an *auxiliary* integrable model such that (1) the eigenvectors for the original model can be expressed as a linear combination of the eigenvectors for the auxiliary model and (2) the coefficients of this decomposition should factorize into products of the single variable functions (phenomenon of 'separation of variables').

Second, the auxiliary problem should be chosen in such a way that the construction of its eigenvectors is a simple iteration process: eigenvectors for the auxiliary model of size n have to be obtained from the eigenvectors for the model of size $n - 1$.

An example of the realization of the first step in the case of the Toda chain model was proposed in the paper [18]. The auxiliary model for the periodic Toda chain was the open Toda chain. In the paper [19] on the Toda example, this approach has been formalized as 'the functional Bethe ansatz' [8]. A complete realization of this step for the periodic Toda chain model can be found in the paper [9].

Regarding a recurrent procedure for eigenvectors of the auxiliary problem, probably the first reference to this possibility may be found in the series of lectures [20]. The main idea of this approach can be formulated as follows. Consider an integrable quantum chain model of size n. The monodromy matrix is the product of n L-operators. We decompose a system into two subsystems of sizes n_1 and n_2 such that $n = n_1 + n_2$. Suppose we can solve the eigenvalue and eigenvector problems for the subsystems. In [20], Sklyanin claims that there is a relation between the eigenvectors of the original system and the eigenvectors of its smaller subsystems. For the open Toda chain model, this Sklyanin approach has been realized in the paper [10].

In our case of the BBS chain model, the auxiliary model is governed by the set of commuting integrals (16). So we first solve the problem of finding the eigenvectors for these integrals. Then we shall show that the eigenvectors for the operators (14) can be constructed as linear combinations of the eigenvectors of the set (16). The multi-variable coefficients of this decomposition admit the separation of variables and can be written as products of single variable functions, each satisfying a Baxter difference equation. We shall obtain this Baxter equation for generic N and solve it explicitly for $N = 2$ corresponding to the free fermion point of the generalized Ising model [16]. Note that the eigenvectors of the commuting set of operators which come from the generating polynomial $A_n(\lambda)$ ($[A_n(\lambda), A_n(\mu)] = 0$) were found in paper [21].

2. Eigenvectors of $B_n(\lambda)$

2.1. Consequences of the RTT relations

We start with the second step of the programme described in section 1.3. Following [8, 10, 20], we first construct eigenvectors of $B_n(\lambda)$. According to (16), any common eigenvector of the commuting set of operators $\mathbf{h}_1, \ldots, \mathbf{h}_n$ is an eigenvector of $B_n(\lambda)$ and the eigenvalue is a polynomial in λ. Factorizing this polynomial we get

$$B_n(\lambda)\Psi_{\boldsymbol{\lambda}} = \lambda\lambda_0 \prod_{k=1}^{n-1}(\lambda - \lambda_k)\Psi_{\boldsymbol{\lambda}}, \qquad \boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}), \tag{22}$$

where we labelled the eigenvector $\Psi_{\boldsymbol{\lambda}}$ by the normalizing factor λ_0 and the $n-1$ non-vanishing zeros $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ of the eigenvalue polynomial.

Now the intertwining relations (13) tell us that if $\Psi_{\boldsymbol{\lambda}}$ is an eigenvector of $B_n(\lambda)$, then by applying repeatedly the operators $A(\lambda_j)$ and $D(\lambda_k)$ $(j, k = 1, \ldots, n-1)$ and $\mathbf{V}_n$ (15), we can generate a whole set of N^n eigenvectors of $B_n(\lambda)$.

The intertwining relations (13) give

$$(\lambda - \omega\mu)A_n(\lambda)B_n(\mu) = \omega(\lambda - \mu)B_n(\mu)A_n(\lambda) + \mu(1-\omega)A_n(\mu)B_n(\lambda), \tag{23}$$

$$(\lambda - \omega\mu)D_n(\mu)B_n(\lambda) = \omega(\lambda - \mu)B_n(\lambda)D_n(\mu) + \lambda(1-\omega)D_n(\lambda)B_n(\mu). \tag{24}$$

Fixing $\lambda = \lambda_k, k = 1, \ldots, n-1$, in (23) and acting by it on $\Psi_{\boldsymbol{\lambda}}$, we obtain

$$B_n(\mu)(A_n(\lambda_k)\Psi_{\boldsymbol{\lambda}}) = \mu\lambda_0(\mu - \omega^{-1}\lambda_k)\prod_{s\neq k}(\mu - \lambda_s)(A_n(\lambda_k)\Psi_{\boldsymbol{\lambda}}). \tag{25}$$

This means that up to a constant

$$A_n(\lambda_k)\Psi_{\boldsymbol{\lambda}} \sim \Psi_{\lambda_0,\ldots,\omega^{-1}\lambda_k,\ldots,\lambda_{n-1}}. \tag{26}$$

Similarly, from (24) we can get

$$D_n(\lambda_k)\Psi_{\boldsymbol{\lambda}} \sim \Psi_{\omega^{-1}\lambda_0,\ldots,\omega\lambda_k,\ldots,\lambda_{n-1}}. \tag{27}$$

(The proportional factors will be obtained later in (70) and (90).) Furthermore, acting by (23) on $\Psi_{\boldsymbol{\lambda}}$ and extracting coefficients of $\lambda^{n+1}\mu^n$ we have

$$\mathbf{V}_n \Psi_{\boldsymbol{\lambda}} \sim \Psi_{\omega^{-1}\lambda_0,\ldots,\lambda_k,\ldots,\lambda_{n-1}}. \tag{28}$$

We see that the operators $A_n(\lambda)$ and $D_n(\lambda)$ at the eigenvalue zeros λ_k of $B_n(\lambda)$, together with the charge rotation operator $\mathbf{V}_n = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_n$, act as cyclic ladder operators on the eigenvectors of $B_n(\lambda)$. So the eigenvalues of $B_n(\lambda)$ can be written as

$$B_n(\lambda)\Psi_{\rho_n} = \lambda r_{n,0}\omega^{-\rho_{n,0}} \prod_{k=1}^{n-1}(\lambda + r_{n,k}\omega^{-\rho_{n,k}})\Psi_{\rho_n}, \tag{29}$$

where $r_{n,s}$, $s = 0,\ldots,n-1$, is a set of constants and we shall use the phases

$$\boldsymbol{\rho}_n = (\rho_{n,0},\ldots,\rho_{n,n-1}) \in (\mathbb{Z}_N)^n \tag{30}$$

as new labels of the eigenvectors. The fact that the eigenvectors of the operator $B_n(\lambda)$ can be considered as dependent only on the integer phases of the roots

$$\lambda_{n,k} = -r_{n,k}\omega^{-\rho_{n,k}} \tag{31}$$

is a common property of the root-of-unity integrable models. The amplitudes $r_{n,k}$ of these roots are fixed by some 'classical' procedure which will be described below. In some cases, this procedure becomes an classical integrable system naturally incorporated into the quantum system (see [29] and references therein).

2.2. *One- and two-site eigenvectors for the auxiliary problem*

We now start to solve the auxiliary problem, which is to compute the eigenvectors of $B_n(\lambda)$ in the basis $\mathcal{V}^{(n)}$. We shall adapt the recursive procedure of Kharchev and Lebedev [10] to the BBS chain model.

In our root-of-unity case, a very important role will be played by the cyclic function $w_p(\gamma)$ [28] which depends on a $\mathbb{Z}_N$-variable γ and on a point $p = (x, y)$ restricted to the Fermat curve $x^N + y^N = 1$. We define $w_p(\gamma)$ by the difference equation

$$\frac{w_p(\gamma)}{w_p(\gamma-1)} = \frac{y}{1-\omega^\gamma x}, \qquad x^N + y^N = 1, \qquad \gamma \in \mathbb{Z}_N, \qquad w_p(0) = 1. \tag{32}$$

The Fermat curve restriction guarantees the cyclic property $w_p(\gamma + N) = w_p(\gamma)$. The function $w_p(\gamma)$ is a root-of-unity analogue of the q-gamma function.

It is convenient to change the bases in the spaces $\mathcal{V}_k$. Instead of $|\gamma\rangle_k$, $\gamma \in \mathbb{Z}_N$, we will use the vectors

$$\psi_\rho^{(k)} = \sum_{\gamma\in\mathbb{Z}_N} w_{p_k}(\gamma-\rho)|\gamma\rangle_k, \qquad \rho \in \mathbb{Z}_N, \tag{33}$$

which are eigenvectors of the upper off-diagonal matrix element $\lambda\mathbf{u}_k^{-1}(a_k - b_k\mathbf{v}_k)$ of the operator L_k:

$$\begin{aligned}
\lambda\mathbf{u}_k^{-1}(a_k - b_k\mathbf{v}_k)\psi_\rho^{(k)} &= \lambda a_k \sum_{\gamma\in\mathbb{Z}_N} w_{p_k}(\gamma-\rho)\omega^{-\gamma}|\gamma\rangle_k - \lambda b_k \sum_{\gamma\in\mathbb{Z}_N} w_{p_k}(\gamma-\rho)\omega^{-(\gamma+1)}|\gamma+1\rangle_k \\
&= \lambda a_k \sum_{\gamma\in\mathbb{Z}_N} w_{p_k}(\gamma-\rho)\omega^{-\gamma}|\gamma\rangle_k - \lambda b_k \sum_{\gamma\in\mathbb{Z}_N} w_{p_k}(\gamma-\rho-1)\omega^{-\gamma}|\gamma\rangle_k \\
&= \lambda \sum_{\gamma\in\mathbb{Z}_N} w_{p_k}(\gamma-\rho)\left[\left(a_k - \frac{b_k}{y_k}\right)\omega^{-\gamma} + b_k\frac{x_k}{y_k}\omega^{-\rho}\right]|\gamma\rangle_k = \lambda r_k\omega^{-\rho}\psi_\rho^{(k)}.
\end{aligned} \tag{34}$$

In the first step we used (8) and to obtain the last line we used (32) with $y_k = b_k/a_k$, $r_k = x_k a_k$. The Fermat curve restriction for $p_k = (x_k, y_k)$ gives $r_k^N = a_k^N - b_k^N$. We see that if $r_k = 0$ (in particular, in the superintegrable case (21)) it leads to $x_k = 0$, $y_k = 1$. In this case, (33) does not give a new basis in $\mathcal{V}_k$. This is the reason why we exclude values of the parameters which lead to the degeneration of the cyclic function $w_p(\gamma)$.

This sequence of operations applied in (34) will be performed rather often in the following derivations. The application of $\mathbf{v}_k$ shifts the spin index. This is compensated by the shift of the summation variable, which results in an opposite shift of the argument of w_p. This in turn is removed using (32).

The operator $\mathbf{v}_k$ shifts the index of $\psi_\rho^{(k)}$:

$$\mathbf{v}_k\psi_\rho^{(k)} = \sum_{\gamma\in\mathbb{Z}_N} w_{p_k}(\gamma-\rho)|\gamma+1\rangle_k = \sum_{\gamma\in\mathbb{Z}_N} w_{p_k}(\gamma-\rho-1)|\gamma\rangle_k = \psi_{\rho+1}^{(k)}. \tag{35}$$

Using (34) for $k = 1$ and comparing to (29), we write the one-site eigenvector as $\Psi_{\rho_{1,0}} := \psi_{\rho_{1,0}}^{(1)}$. With $r_{1,0} = r_1$, we have

$$B_1(\lambda)\Psi_{\rho_{1,0}} = \lambda r_{1,0}\omega^{-\rho_{1,0}}\Psi_{\rho_{1,0}}, \qquad A_1(\lambda)\Psi_{\rho_{1,0}} = \Psi_{\rho_{1,0}} + \lambda\varkappa_1\Psi_{\rho_{1,0}+1}. \tag{36}$$

The construction of the two-site eigenvectors $\Psi_{\boldsymbol{\rho}_2}$ will show us the first step of the recursive method. In accordance with (29), we are looking for eigenvectors $\Psi_{\boldsymbol{\rho}_2}$ ($\boldsymbol{\rho}_2 \equiv (\rho_{2,0}, \rho_{2,1}) \in \mathbb{Z}_N\times\mathbb{Z}_N$) of the two-site operator $B_2(\lambda)$, which should satisfy

$$B_2(\lambda)\Psi_{\boldsymbol{\rho}_2} = \lambda r_{2,0}\omega^{-\rho_{2,0}}(\lambda + r_{2,1}\omega^{-\rho_{2,1}})\Psi_{\boldsymbol{\rho}_2}. \tag{37}$$

We suppose that $\Psi_{\boldsymbol{\rho}_2}$ can be written as a linear combinations of products of one-site eigenvectors

$$\Psi_{\boldsymbol{\rho}_2} = \sum_{\rho_1,\rho_2\in\mathbb{Z}_N} Q(\rho_1,\rho_2|\boldsymbol{\rho}_2)\psi_{\rho_1}^{(1)}\otimes\psi_{\rho_2}^{(2)}. \tag{38}$$

Using (36) and (34), the matrix Q can be calculated as follows:

$$\begin{aligned} B_2(\lambda)\Psi_{\boldsymbol{\rho}_2} &= \left(A_1(\lambda)\lambda\mathbf{u}_2^{-1}(a_2 - b_2\mathbf{v}_2) + B_1(\lambda)(\lambda a_2c_2 + b_2d_2\mathbf{v}_2/\varkappa_2)\right)\Psi_{\boldsymbol{\rho}_2} \\ &= \sum_{\rho_1,\rho_2}\left\{Q(\rho_1,\rho_2|\boldsymbol{\rho}_2)(\lambda r_2\omega^{-\rho_2} + \lambda^2 a_2c_2r_1\omega^{-\rho_1}) + Q(\rho_1-1,\rho_2|\boldsymbol{\rho}_2)\lambda^2\varkappa_1r_2\omega^{-\rho_2}\right. \\ &\quad \left. + Q(\rho_1,\rho_2-1|\boldsymbol{\rho}_2)\frac{b_2d_2}{\varkappa_2}\lambda r_1\omega^{-\rho_1}\right\}\psi_{\rho_1}^{(1)}\otimes\psi_{\rho_2}^{(2)}. \end{aligned} \tag{39}$$

Comparing powers of the spectral parameter λ in (39) and in (37), together with (38), we get

$$(r_{2,0}\omega^{-\rho_{2,0}} - a_2c_2r_1\omega^{-\rho_1})Q(\rho_1,\rho_2|\boldsymbol{\rho}_2) = \varkappa_1r_2\omega^{-\rho_2}Q(\rho_1-1,\rho_2|\boldsymbol{\rho}_2), \tag{40}$$

$$(r_{2,0}r_{2,1}\omega^{-\rho_{2,0}-\rho_{2,1}} - r_2\omega^{-\rho_2})Q(\rho_1,\rho_2|\boldsymbol{\rho}_2) = \frac{b_2d_2}{\varkappa_2}r_1\omega^{-\rho_1}Q(\rho_1,\rho_2-1|\boldsymbol{\rho}_2). \tag{41}$$

The difference equations (40) and (41) have the solution

$$Q(\rho_1,\rho_2|\boldsymbol{\rho}_2) = \frac{\omega^{-(\rho_{2,0}+\rho_{2,1}-\rho_1)(\rho_{2,0}-\rho_2)}}{w_{p_{2,0}}(\rho_{2,0}-\rho_1-1)w_{\tilde{p}_2}(\rho_{2,0}+\rho_{2,1}-\rho_2-1)}, \tag{42}$$

where $p_{2,0} = (x_{2,0}, y_{2,0})$, $\tilde{p}_2 = (\tilde{x}_2, \tilde{y}_2)$ and

$$x_{2,0} = a_2c_2\frac{r_1}{r_{2,0}}, \qquad y_{2,0} = \varkappa_1\frac{r_2}{r_{2,0}}, \qquad \tilde{x}_2 = \frac{r_2}{r_{2,0}r_{2,1}}, \qquad \tilde{y}_2 = \frac{b_2d_2}{\varkappa_2}\frac{r_1}{r_{2,0}r_{2,1}}. \tag{43}$$

The parameters $r_{2,0}$ and $r_{2,1}$ are determined from the condition that the points $p_{2,0}$ and $\tilde{p}_2$ belong to the Fermat curve.

One can proceed this way to construct n-site eigenvectors of the auxiliary problem. In fact, in order to see the general structure emerging, one has to go to the four-site case. We shall not do this here, but rather in section 3 we shall prove the general result by induction. This proof will use recursive relations between amplitudes $r_{n,k}$, $k = 0, 1, \ldots, n-1$, formulated in subsection 2.4. Fermat parameters $p = (x, y)$ of the cyclic functions $w_p(\rho)$ appearing in our construction will depend on these amplitudes. Compatibility conditions between recursive relations for the amplitudes and the Fermat curve equation $x^N + y^N = 1$ can be formulated as a 'classical' BBS chain model [29]. This model will be formulated in the next subsection using an averaging procedure for the cyclic L-operators (6) given in [14].

2.3. Determination of the parameters $r_{m,s}$

Let us define the 'classical' counterpart $\mathcal{O}(\lambda^N)$ of a quantum cyclic operator $O(\lambda)$ using the averaging procedure [14]:

$$\mathcal{O}(\lambda^N) = \langle O\rangle(\lambda^N) = \prod_{s\in\mathbb{Z}_N} O(\omega^s\lambda) \tag{44}$$

and apply this procedure to the entries of the quantum L-operator (6). Denote the result by $\mathcal{L}_k(\lambda^N)$

$$\mathcal{L}_k(\lambda^N) = \begin{pmatrix} \langle L_{00}\rangle & \langle L_{01}\rangle \\ \langle L_{10}\rangle & \langle L_{11}\rangle \end{pmatrix} = \begin{pmatrix} 1 - \epsilon\varkappa_k^N\lambda^N & -\epsilon\lambda^N(a_k^N - b_k^N) \\ c_k^N - d_k^N & b_k^N d_k^N/\varkappa_k^N - \epsilon\lambda^N a_k^N c_k^N \end{pmatrix}, \tag{45}$$

where $\epsilon = (-1)^N$, and call it as the 'classical' $\mathcal{L}$-operator of the classical BBS model. Accordingly, the classical monodromy $\mathcal{T}_m$ for the m-site chain is

$$\mathcal{T}_m(\lambda^N) = \mathcal{L}_1(\lambda^N)\mathcal{L}_2(\lambda^N)\cdots\mathcal{L}_m(\lambda^N) = \begin{pmatrix} \mathcal{A}_m(\lambda^N) & \mathcal{B}_m(\lambda^N) \\ \mathcal{C}_m(\lambda^N) & \mathcal{D}_m(\lambda^N) \end{pmatrix}, \tag{46}$$

where the entries are polynomials of λ^N. By proposition 1.5 of [14], these polynomials coincide with averages $\langle A_m\rangle$, $\langle B_m\rangle$, $\langle C_m\rangle$ and $\langle D_m\rangle$ of the entries of (10). This proposition provides a tool for finding the Nth powers of the amplitudes $r_{m,s}$: applying (44) to (29) we obtain

$$\mathcal{B}_m(\lambda^N) = (-\epsilon)^m\lambda^N r_{m,0}^N \prod_{s=1}^{m-1} \left(\lambda^N - \epsilon r_{m,s}^N\right). \tag{47}$$

This relation together with (45) and (46) allows us to find $r_{m,s}^N$ in terms of the parameters $a_k^N, b_k^N, c_k^N, d_k^N$ and $\varkappa_k^N$, $k = 1, \ldots, m$. The problem of finding the amplitudes $r_{m,s}$ is reduced to the problem of solving a $(m-1)$th degree algebraic relation. As shown in the appendix, in the case of the homogeneous BBS chain model the problem is reduced to solving a quadratic equation only. The described procedure gives the amplitudes $r_{m,s}$ up to some roots of unity. In fact, we can fix these phases arbitrarily because this leads just to relabelling of the eigenvectors. In what follows, we suppose that we fixed some solution $\{r_{m,s}\}$ in terms of the parameters $a_k^N, b_k^N, c_k^N, d_k^N$ and $\varkappa_k^N$.

Let us give a recursive description for $\mathcal{B}_m(\lambda^N)$. From (46), we immediately read off the recursion relations

$$\begin{aligned} \mathcal{A}_m(\lambda^N) &= \left(1 - \epsilon\varkappa_m^N\lambda^N\right)\mathcal{A}_{m-1}(\lambda^N) + \left(c_m^N - d_m^N\right)\mathcal{B}_{m-1}(\lambda^N), \\ \mathcal{B}_m(\lambda^N) &= -\epsilon\lambda^N\left(a_m^N - b_m^N\right)\mathcal{A}_{m-1}(\lambda^N) + \left(b_m^N d_m^N/\varkappa_m^N - \epsilon\lambda^N a_m^N c_m^N\right)\mathcal{B}_{m-1}(\lambda^N). \end{aligned} \tag{48}$$

Combining these two relations we get

$$\mathcal{A}_m(\lambda^N) = \frac{\epsilon \varkappa_m^N \lambda^N - 1}{\epsilon \lambda^N \left(a_m^N - b_m^N\right)} \mathcal{B}_m(\lambda^N) + \frac{\det \mathcal{L}_m(\lambda^N)}{\epsilon \lambda^N \left(a_m^N - b_m^N\right)} \mathcal{B}_{m-1}(\lambda^N), \tag{49}$$

where

$$\det \mathcal{L}_m(\lambda^N) = \left(d_m^N - \epsilon \lambda^N c_m^N \varkappa_m^N\right)\left(b_m^N - \epsilon \lambda^N a_m^N \varkappa_m^N\right) / \varkappa_m^N. \tag{50}$$

Substituting $\mathcal{A}_{m-1}$ from this equation with m replaced by $m-1$ into (48), we obtain a three-term recursion for $\mathcal{B}_m(\lambda^N)$:

$$\begin{aligned} \mathcal{B}_m(\lambda^N) = {} & \left(\frac{r_m^N}{r_{m-1}^N} \left(1 - \epsilon \varkappa_{m-1}^N \lambda^N\right) + b_m^N d_m^N / \varkappa_m^N - \epsilon \lambda^N a_m^N c_m^N \right) \mathcal{B}_{m-1}(\lambda^N) \\ & + \frac{r_m^N}{r_{m-1}^N \varkappa_{m-1}^N} \left(b_{m-1}^N - \epsilon \lambda^N a_{m-1}^N \varkappa_{m-1}^N\right)\left(\epsilon \lambda^N c_{m-1}^N \varkappa_{m-1}^N - d_{m-1}^N\right) \mathcal{B}_{m-2}(\lambda^N), \qquad m \geqslant 2, \end{aligned} \tag{51}$$

where we abbreviated

$$r_m^N = a_m^N - b_m^N. \tag{52}$$

To define $\mathcal{B}_m(\lambda^N)$ by (51) we have to provide the initial values

$$\mathcal{B}_1(\lambda^N) = -\epsilon \lambda^N r_1^N, \qquad \mathcal{B}_0(\lambda^N) = 0.$$

2.4. *Fermat curve points appearing in the construction of the eigenvectors of $B_n(\lambda)$*

As we have seen in the case of the two-site chain, formulae (38), (42) for the eigenvectors are given in terms of the points $p_{2,0}$ and $\tilde{p}_2$ on the Fermat curve. The coordinates of these points are fixed by the values of amplitudes $r_{2,0}, r_{2,1}$, see (43). In the n-site case, four types of such points will appear:

$$\begin{aligned} \tilde{p}_m &= (\tilde{x}_m, \tilde{y}_m), \qquad & p_{m,s} &= (x_{m,s}, y_{m,s}), \\ \tilde{p}_{m,s} &= (\tilde{x}_{m,s}, \tilde{y}_{m,s}), \qquad & p_{m',s'}^{m,s} &= \left(x_{m',s'}^{m,s}, y_{m',s'}^{m,s}\right). \end{aligned} \tag{53}$$

The coordinates of these points are expressed in terms of amplitudes $r_{m,s}$, $m = 1, \ldots, n$, $s = 0, \ldots, m-1$ (defined as *some* solutions of equations (47), $m = 1, \ldots, n$) by

$$x_{m',s'}^{m,s} = r_{m,s}/r_{m',s'}, \qquad x_{m,s} = a_m \varkappa_m r_{m,s}/b_m, \qquad \tilde{x}_{m,s} = d_m/(\varkappa_m c_m r_{m,s}), \qquad s, s' \geqslant 1. \tag{54}$$

The corresponding $y_{m',s'}^{m,s}$, $\tilde{y}_{m,s}$ are defined by the only condition on $p_{m',s'}^{m,s}$, $\tilde{p}_{m,s}$ to belong to the Fermat curve. The coordinates $y_{m-1,l}$, $1 \leqslant l \leqslant m-2$, are defined by

$$\frac{\tilde{r}_{m-1} r_{m,0} r_{m-1}}{\tilde{r}_{m-2} r_{m-1,0} r_m b_{m-1} c_{m-1} y_{m-1,l} \tilde{y}_{m-1,l}} \prod_{s \neq l}^{m-2} \frac{y_{m-1,s}^{m-1,l}}{y_{m-1,l}^{m-1,s}} \frac{\prod_{k=1}^{m-1} y_{m-1,l}^{m,k}}{\prod_{s=1}^{m-3} y_{m-2,s}^{m-1,l}} = 1, \qquad l = 1, \ldots, m-2, \tag{55}$$

where

$$\tilde{r}_m = r_{m,0} r_{m,1} \ldots r_{m,m-1}. \tag{56}$$

The coordinates of the points $p_{m,0}$ and $\tilde{p}_m$ are defined by

$$x_{m,0} r_{m,0} = r_{m-1,0} a_m c_m, \qquad y_{m,0} r_{m,0} = \varkappa_1 \varkappa_2 \cdots \varkappa_{m-1} r_m, \tag{57}$$

$$\tilde{x}_m \tilde{r}_m = r_m, \qquad \tilde{y}_m \tilde{r}_m = b_m d_m \tilde{r}_{m-1}/\varkappa_m. \tag{58}$$

Formulae (55)–(58) are result from the solution of the eigenvector problem (29), see section 3.

The condition on the points $p_{m-1,l}$ $(1 \leqslant l \leqslant m-2)$, $p_{m,0}$ and $\tilde{p}_m$ defined by (55), (57), (58) to belong to the Fermat curve gives

$$r_{m-1}^N \varkappa_{m-1}^N r_{m,0}^N \prod_{k=1}^{m-1} \left(r_{m-1,l}^N - r_{m,k}^N\right) = r_m^N r_{m-2,0}^N \left(b_{m-1}^N - a_{m-1}^N \varkappa_{m-1}^N r_{m-1,l}^N\right) \\ \times \left(r_{m-1,l}^N c_{m-1}^N \varkappa_{m-1}^N - d_{m-1}^N\right) \prod_{s=1}^{m-3} \left(r_{m-1,l}^N - r_{m-2,s}^N\right), \qquad l = 1, 2, \ldots, m-2, \tag{59}$$

$$r_{m,0}^N = r_{m-1,0}^N a_m^N c_m^N + \varkappa_1^N \varkappa_2^N \cdots \varkappa_{m-1}^N r_m^N, \tag{60}$$

$$\tilde{r}_m^N \equiv r_{m,0}^N r_{m,1}^N \cdots r_{m,m-1}^N = r_m^N + b_m^N d_m^N \tilde{r}_{m-1}^N / \varkappa_m^N. \tag{61}$$

In order to relate these relations to the recurrent formulae of the classical BBS model (51), we observe that relations (60) (resp. (61)) follow from the relations obtained by the consideration of the highest (resp. lowest) terms in λ in (51) starting from $m = 2$. Then, fixing in (51) λ^N successively at the $m-2$ non-vanishing zeros of $\mathcal{B}_{m-1}$, i.e. putting $\lambda^N = \epsilon r_{m-1,l}^N, l = 1, \ldots, m-2$, we obtain (59). Thus, the points $p_{m-1,l}$, $p_{m,0}$ and $\tilde{p}_m$ defined by (55), (57), (58) belong to the Fermat curve automatically.

At the end of this section, we would like to mention that the amplitudes $r_{m,s}$ can be found directly (i.e. not using the results from the previous subsection) from relations (55), (57), (58) considered as equations with respect to $r_{m,s}$ and the coordinates of the Fermat curve points (53). These equations can be solved recursively starting from $m = 2$ and taking Nth powers of these relations (see (59)–(61)). Then the explicit formulae for the eigenvectors from the next section allow us to obtain the Tarasov proposition 1.5 in [14] as a corollary.

3. Inductive proof of the general solution of the auxiliary problem

Recall from (36) that the vector $\Psi_{\rho_{1,0}} := \psi_{\rho_{1,0}}^{(1)} \in \mathcal{V}_1$ is an eigenvector for $B_1(\lambda)$:

$$B_1(\lambda)\Psi_{\rho_{1,0}} = \lambda r_{1,0} \omega^{-\rho_{1,0}} \Psi_{\rho_{1,0}},$$

and recall from (29), (30) that the eigenvectors Ψ_{ρ_n} of $B_n(\lambda)$ were labelled by the vector $\boldsymbol{\rho}_n = (\rho_{n,0}, \ldots, \rho_{n,n-1}) \in (\mathbb{Z}_N)^n$. Let us further define

$$\tilde{\rho}_n = \sum_{k=0}^{n-1} \rho_{n,k}, \qquad \boldsymbol{\rho}_n' = (\rho_{n,1}, \ldots, \rho_{n,n-1}) \in (\mathbb{Z}_N)^{n-1}. \tag{62}$$

$\boldsymbol{\rho}_n^{\pm k}$ denotes the vector $\boldsymbol{\rho}_n$ in which $\rho_{n,k}$ is replaced by $\rho_{n,k} \pm 1$, i.e.

$$\boldsymbol{\rho}_n^{\pm k} = (\rho_{n,0}, \ldots, \rho_{n,k} \pm 1, \ldots, \rho_{n,n-1}), \qquad k = 0, 1, \ldots, n.$$

Theorem 1 gives a procedure to obtain the eigenvectors $\Psi_{\rho_n} \in \mathcal{V}^{(n)}$, $n \geqslant 2$, of $B_n(\lambda)$ from eigenvectors $\Psi_{\rho_{n-1}} \in \mathcal{V}^{(n-1)}$ of $B_{n-1}(\lambda)$ and single-site vectors $\psi_{\rho_n}^{(n)} \in \mathcal{V}_n$ defined by (33). We start from $\Psi_{\rho_{1,0}}$. As a result of the first step of the induction, we obtain the two-site result (38) with (42), (43).

The following theorem is valid provided $r_m^N \neq 0$, the polynomials $\mathcal{B}_m(\lambda^N)/\lambda^N$, $m = 2, \ldots, n$, have nonzero simple zeros and $\det \mathcal{T}_n\left(\epsilon r_{m,s}^N\right) \neq 0$ (cf the definition of the B-representation in [14]).

Theorem 1. *The vector*

$$\Psi_{\rho_n} = \sum_{\substack{\rho_{n-1}\in(\mathbb{Z}_N)^{n-1} \\ \rho_n\in\mathbb{Z}_N}} Q(\boldsymbol{\rho}_{n-1}, \rho_n|\boldsymbol{\rho}_n)\Psi_{\rho_{n-1}} \otimes \psi_{\rho_n}^{(n)}, \tag{63}$$

where

$$Q(\boldsymbol{\rho}_{n-1}, \rho_n|\boldsymbol{\rho}_n) = \frac{\omega^{(\tilde{p}_n-\tilde{p}_{n-1})(\rho_n-\rho_{n,0})}}{w_{p_{n0}}(\rho_{n,0}-\rho_{n-1,0}-1)w_{\tilde{p}_n}(\tilde{\rho}_n-\rho_n-1)} \times \frac{\prod_{l=1}^{n-2}\prod_{k=1}^{n-1} w_{p_{n-1,l}^{n,k}}(\rho_{n-1,l}-\rho_{n,k})}{\prod_{\substack{j,l=1\\(j\neq l)}}^{n-2} w_{p_{n-1,j}^{n-1,l}}(\rho_{n-1,j}-\rho_{n-1,l})} \prod_{l=1}^{n-2} \frac{w_{p_{n-1,l}}(-\rho_{n-1,l})}{w_{\tilde{p}_{n-1,l}}(\rho_{n-1,l})} \tag{64}$$

is an eigenvector of $B_n(\lambda)$:

$$B_n(\lambda)\Psi_{\rho_n} = \lambda r_{n,0}\omega^{-\rho_{n,0}} \prod_{k=1}^{n-1}(\lambda + r_{n,k}\omega^{-\rho_{n,k}})\Psi_{\rho_n}. \tag{65}$$

The Fermat curve points $\tilde{p}_n$, $p_{n,l}$, $\tilde{p}_{n,l}$, $p_{n',l}^{n,k}$ and $r_{n,k}$, entering (64) are related to the parameters of the model $a_s, b_s, c_s, d_s, \varkappa_s$ by equations (54), (55), (57), (58).

$A_n(\lambda)$ acts on Ψ_{ρ_n} as follows:

$$A_n(\lambda)\Psi_{\rho_n} = \prod_{s=1}^{n-1}\left(1-\frac{\lambda}{\lambda_{n,s}}\right)\Psi_{\rho_n} + \lambda\varkappa_1\cdots\varkappa_n\prod_{s=1}^{n-1}(\lambda-\lambda_{n,s})\Psi_{\rho_n^{+0}} + \sum_{k=1}^{n-1}\left(\prod_{s\neq k}\frac{\lambda-\lambda_{n,s}}{\lambda_{n,k}-\lambda_{n,s}}\right)\frac{\lambda}{\lambda_{n,k}}\varphi_k(\boldsymbol{\rho}_n')\Psi_{\rho_n^{+k}}, \tag{66}$$

where

$$\varphi_k(\boldsymbol{\rho}_n') = -\frac{\tilde{r}_{n-1}}{r_n}\omega^{-\tilde{p}_n+\rho_{n,0}}F_n(\lambda_{n,k}/\omega)\prod_{s=1}^{n-2}y_{n-1,s}^{n,k} \tag{67}$$

with

$$F_n(\lambda) = (b_n + \omega a_n\varkappa_n\lambda)(\lambda c_n + d_n/\varkappa_n). \tag{68}$$

Corollary. *In particular, at the $n-1$ zeros $\lambda_{n,k}$ of the eigenvalue polynomial of $B_n(\lambda)$*

$$\lambda_{n,k} = -r_{n,k}\omega^{-\rho_{n,k}}, \qquad k = 1, \ldots, n-1, \tag{69}$$

the operator A_n acts as a shift operator for the kth index of Ψ_{ρ_n}:

$$A_n(\lambda_{n,k})\Psi_{\rho_n} = \varphi_k(\boldsymbol{\rho}_n')\Psi_{\rho_n^{+k}}. \tag{70}$$

Further, the term in (66) of highest degree in λ gives: $\mathbf{V}_n = \mathbf{v}_1\mathbf{v}_2\ldots\mathbf{v}_n$ is a shift operator for the zeroth index of Ψ_{ρ_n}:

$$\mathbf{V}_n\Psi_{\rho_n} = \Psi_{\rho_n^{+0}}. \tag{71}$$

Proof. We shall prove theorem 1 by induction, showing that if it is valid for $n-1$ site eigenvectors, then it follows for n site eigenvectors. Namely, we assume the correctness of the following formulae:

$$B_{n-1}(\lambda)\Psi_{\rho_{n-1}} = \lambda r_{n-1,0}\omega^{-\rho_{n-1,0}}\prod_{l=1}^{n-2}(\lambda-\lambda_{n-1,l})\Psi_{\rho_{n-1}}, \tag{72}$$

$$\begin{aligned}A_{n-1}(\lambda)\Psi_{\rho_{n-1}} &= \sum_{l=1}^{n-2}\left(\prod_{s\neq l}\frac{\lambda-\lambda_{n-1,s}}{\lambda_{n-1,l}-\lambda_{n-1,s}}\right)\frac{\lambda}{\lambda_{n-1,l}}\varphi_l(\boldsymbol{\rho}'_{n-1})\Psi_{\rho^{+l}_{n-1}}\\ &\quad+\prod_{s=1}^{n-2}\left(1-\frac{\lambda}{\lambda_{n-1,s}}\right)\Psi_{\rho_{n-1}}+\lambda\varkappa_1\cdots\varkappa_{n-1}\prod_{s=1}^{n-2}(\lambda-\lambda_{n-1,s})\Psi_{\rho^{+0}_{n-1}},\end{aligned} \tag{73}$$

where $\lambda_{n-1,l} = -r_{n-1,l}\omega^{-\rho_{n-1,l}}$ and the formulae for $\varphi_l(\boldsymbol{\rho}'_{n-1})$ are given by (67) with n replaced by $n-1$.

Formula (65) for $B_n(\lambda)\Psi_{\rho_n}$.

In order to prove the eigenvalue formula (65), we use the following relation:

$$B_n(\lambda) = A_{n-1}(\lambda)\lambda\mathbf{u}_n^{-1}(a_n-b_n\mathbf{v}_n)+B_{n-1}(\lambda)\left(\lambda a_n c_n+\frac{b_n d_n}{\varkappa_n}\mathbf{v}_n\right), \tag{74}$$

which follows directly from (6) and (10). We apply its left-hand side to the left-hand side of (63) and its right-hand side to the right-hand side of (63). On the right, we use (72), (73), (34), (35). According to (73), A_{n-1} introduces shifts in the indices $\boldsymbol{\rho}_{n-1}$ of $\Psi_{\rho_{n-1}}$, while the second term involving $\mathbf{v}_n$ shifts the index of $\psi^{(n)}_{\rho_n}$. Since we are looking for an eigenstate, by shifting the summation indices we restore the original indices. However, this leaves a change in the matrix $Q(\boldsymbol{\rho}_{n-1},\rho_n|\boldsymbol{\rho}_n)$. Now the difference equation (32) for the $w_p(\gamma)$ functions appearing in $Q(\boldsymbol{\rho}_{n-1},\rho_n|\boldsymbol{\rho}_n)$ is used, producing several factors under the summation which together we call R:

$$B_n(\lambda)\Psi_{\rho_n} = \sum_{\substack{\rho_{n-1}\in(\mathbb{Z}_N)^{n-1}\\ \rho_n\in\mathbb{Z}_N}} Q(\boldsymbol{\rho}_{n-1},\rho_n|\boldsymbol{\rho}_n)R\Psi_{\rho_{n-1}}\otimes\psi^{(n)}_{\rho_n}.$$

After some calculation, we obtain

$$\begin{aligned}R &= \left\{\sum_{l=1}^{n-2}\left(\prod_{s\neq l}\frac{\lambda-\lambda_{n-1,s}}{\omega\lambda_{n-1,l}-\lambda_{n-1,s}}\right)\frac{\lambda}{\omega\lambda_{n-1,l}}\varphi_l\left(\boldsymbol{\rho}'^{-l}_{n-1}\right)\frac{Q\left(\boldsymbol{\rho}^{-l}_{n-1},\rho_n\middle|\boldsymbol{\rho}_n\right)}{Q(\boldsymbol{\rho}_{n-1},\rho_n|\boldsymbol{\rho}_n)}\right.\\ &\quad\left.+\prod_{s=1}^{n-2}\left(1-\frac{\lambda}{\lambda_{n-1,s}}\right)+\lambda\varkappa_1\cdots\varkappa_{n-1}\prod_{s=1}^{n-2}(\lambda-\lambda_{n-1,s})\frac{Q\left(\boldsymbol{\rho}^{-0}_{n-1},\rho_n\middle|\boldsymbol{\rho}_n\right)}{Q(\boldsymbol{\rho}_{n-1},\rho_n|\boldsymbol{\rho}_n)}\right\}\lambda r_n\omega^{-\rho_n}\\ &\quad+\lambda r_{n-1,0}\omega^{-\rho_{n-1,0}}\prod_{l=1}^{n-2}(\lambda-\lambda_{n-1,l})\left(\lambda a_n c_n+\frac{b_n d_n}{\varkappa_n}\frac{Q(\boldsymbol{\rho}_{n-1},\rho_n-1|\boldsymbol{\rho}_n)}{Q(\boldsymbol{\rho}_{n-1},\rho_n|\boldsymbol{\rho}_n)}\right).\end{aligned}$$

We have to show that

$$R = \lambda r_{n,0}\omega^{-\rho_{n,0}}\prod_{k=1}^{n-1}(\lambda-\lambda_{n,k}),\qquad \lambda_{n,k} = -r_{n,k}\omega^{-\rho_{n,k}}. \tag{75}$$

This will prove (65). Using definitions (54) of $x_{m,s}$ and $\tilde{x}_{m,s}$, we can rewrite (68) for the argument $\lambda = \lambda_{n,k}/\omega$ as follows:

$$F_n(\lambda_{n,k}/\omega) = \lambda_{n,k}b_n c_n\omega^{-1}\left(1-x_{n,k}\omega^{-\rho_{n,k}}\right)\left(1-\tilde{x}_{n,k}\omega^{\rho_{n,k}+1}\right). \tag{76}$$

Taking into account expression (64) for $Q(\boldsymbol{\rho}_{n-1}, \rho_n|\boldsymbol{\rho}_n)$, the definition for $w_p(\gamma)$ and relations (54)–(57), (76), we obtain

$$\frac{Q(\boldsymbol{\rho}_{n-1}^{-l}, \rho_n|\boldsymbol{\rho}_n)}{Q(\boldsymbol{\rho}_{n-1}, \rho_n|\boldsymbol{\rho}_n)} = \omega^{\rho_n - \rho_{n,0}} \prod_{k=1}^{n-1} \frac{w_{p_{n-1,l}^{n,k}}(\rho_{n-1,l} - \rho_{n,k} - 1)}{w_{p_{n-1,l}^{n,k}}(\rho_{n-1,l} - \rho_{n,k})} \frac{w_{p_{n-1,l}}(-\rho_{n-1,l} + 1)}{w_{p_{n-1,l}}(-\rho_{n-1,l})}$$
$$\times \frac{w_{\tilde{p}_{n-1,l}}(\rho_{n-1,l})}{w_{\tilde{p}_{n-1,l}}(\rho_{n-1,l} - 1)} \prod_{s\neq l} \left(\frac{w_{p_{n-1,s}^{n-1,l}}(\rho_{n-1,s} - \rho_{n-1,l})}{w_{p_{n-1,s}^{n-1,l}}(\rho_{n-1,s} - \rho_{n-1,l} + 1)} \frac{w_{p_{n-1,l}^{n-1,s}}(\rho_{n-1,l} - \rho_{n-1,s})}{w_{p_{n-1,l}^{n-1,s}}(\rho_{n-1,l} - \rho_{n-1,s} - 1)} \right)$$
$$= \frac{\omega}{\varphi_l(\boldsymbol{\rho}'^{-l}_{n-1})} \frac{r_{n,0}\omega^{-\rho_{n,0}}}{r_n \omega^{-\rho_n}} \prod_{k=1}^{n-1} (\lambda_{n-1,l} - \lambda_{n,k}) \prod_{s\neq l} \frac{\omega\lambda_{n-1,l} - \lambda_{n-1,s}}{\lambda_{n-1,l} - \lambda_{n-1,s}},$$
$$\frac{Q(\boldsymbol{\rho}_{n-1}^{-0}, \rho_n|\boldsymbol{\rho}_n)}{Q(\boldsymbol{\rho}_{n-1}, \rho_n|\boldsymbol{\rho}_n)} = \omega^{\rho_n - \rho_{n,0}} \frac{w_{p_{n0}}(\rho_{n,0} - \rho_{n-1,0} - 1)}{w_{p_{n0}}(\rho_{n,0} - \rho_{n-1,0})} = \frac{r_{n,0}\omega^{-\rho_{n,0}} - r_{n-1,0}\omega^{-\rho_{n-1,0}} a_n c_n}{\varkappa_1 \varkappa_2 \cdots \varkappa_{n-1} r_n \omega^{-\rho_n}},$$
$$\frac{Q(\boldsymbol{\rho}_{n-1}, \rho_n - 1|\boldsymbol{\rho}_n)}{Q(\boldsymbol{\rho}_{n-1}, \rho_n|\boldsymbol{\rho}_n)} = \omega^{\tilde{\rho}_{n-1} - \tilde{\rho}_n} \frac{w_{\tilde{p}_n}(\tilde{\rho}_n - \rho_n - 1)}{w_{\tilde{p}_n}(\tilde{\rho}_n - \rho_n)} = \frac{\varkappa_n}{b_n d_n} \frac{\tilde{r}_n \omega^{-\tilde{\rho}_n} - r_n \omega^{-\rho_n}}{\tilde{r}_{n-1}\omega^{-\tilde{\rho}_{n-1}}}.$$

Substituting these expressions into R gives

$$R = \left\{ \sum_{l=1}^{n-2} \left(\prod_{s\neq l} \frac{\lambda - \lambda_{n-1,s}}{\lambda_{n-1,l} - \lambda_{n-1,s}} \right) \frac{\lambda}{\lambda_{n-1,l}} \frac{r_{n,0}\omega^{-\rho_{n,0}}}{r_n\omega^{-\rho_n}} \prod_{k=1}^{n-1} (\lambda_{n-1,l} - \lambda_{n,k}) \right.$$
$$\left. + \prod_{s=1}^{n-2} \left(1 - \frac{\lambda}{\lambda_{n-1,s}}\right) + \lambda \prod_{s=1}^{n-2} (\lambda - \lambda_{n-1,s}) \frac{r_{n,0}\omega^{-\rho_{n,0}} - r_{n-1,0}\omega^{-\rho_{n-1,0}} a_n c_n}{r_n \omega^{-\rho_n}} \right\} \lambda r_n \omega^{-\rho_n}$$
$$+ \lambda r_{n-1,0}\omega^{-\rho_{n-1,0}} \prod_{l=1}^{n-2} (\lambda - \lambda_{n-1,l}) \left(\lambda a_n c_n + \frac{\tilde{r}_n \omega^{-\tilde{\rho}_n} - r_n \omega^{-\rho_n}}{\tilde{r}_{n-1}\omega^{-\tilde{\rho}_{n-1}}} \right).$$

After appropriate cancellations, this becomes

$$R = \sum_{l=1}^{n-2} \left(\prod_{s\neq l} \frac{\lambda - \lambda_{n-1,s}}{\lambda_{n-1,l} - \lambda_{n-1,s}} \right) \frac{\lambda^2}{\lambda_{n-1,l}} r_{n,0}\omega^{-\rho_{n,0}} \prod_{k=1}^{n-1} (\lambda_{n-1,l} - \lambda_{n,k})$$
$$+ \lambda^2 \prod_{s=1}^{n-2} (\lambda - \lambda_{n-1,s}) r_{n,0}\omega^{-\rho_{n,0}} + \lambda \tilde{r}_n \omega^{-\tilde{\rho}_n} \prod_{l=1}^{n-2} \left(1 - \frac{\lambda}{\lambda_{n-1,l}}\right). \tag{77}$$

To prove (75) we note that the coefficients at λ^n in both expressions (75) and (77) are $r_{n,0}\omega^{-\rho_{n,0}}$ and coefficients at λ also coincide being $\tilde{r}_n\omega^{-\tilde{\rho}_n}$. Therefore, the difference of these two expressions has the form $\lambda^2 P(\lambda)$ where $P(\lambda)$ is a polynomial of degree $n-3$. Using the explicit expressions (75) and (77), we convince ourselves that $P(\lambda_{n-1,j}) = 0$ and therefore $P(\lambda) \equiv 0$. This completes the proof that $\Psi_{\boldsymbol{\rho}_n}$ defined by (63), (64) is an eigenvector of $B_n(\lambda)$ with eigenvalue (75).

Formula (70) for $A_n(\lambda_{n,k})\Psi_{\boldsymbol{\rho}_n}$.

Next we show the validity of (70), (67). We will need the relation

$$\mathbf{u}_n^{-1}(a_n - b_n \mathbf{v}_n) A_n(\lambda) = (1 + \lambda \varkappa_n \omega^{-1} \mathbf{v}_n) B_n(\lambda)/\lambda - \mathbf{v}_n F_n(\lambda/\omega) B_{n-1}(\lambda)/\lambda, \tag{78}$$

which can be obtained by eliminating A_{n-1} between (74) and

$$A_n(\lambda) = (1 + \lambda \varkappa_n \mathbf{v}_n) A_{n-1}(\lambda) + \mathbf{u}_n (c_n - d_n \mathbf{v}_n) B_{n-1}(\lambda). \tag{79}$$

Let us apply (78) to Ψ_{ρ_n} for $\lambda = -r_{n,k}\omega^{-\rho_{n,k}}$, i.e. at the zeros of $B_n(\lambda)$. This gives

$$\mathbf{u}_n^{-1}(a_n - b_n\mathbf{v}_n)A_n(\lambda_{n,k})\Psi_{\rho_n} = -F_n(\lambda_{n,k}/\omega)/\lambda_{n,k} \times \sum_{\substack{\rho_{n-1}\in(\mathbb{Z}_N)^{n-1}\\ \rho_n\in\mathbb{Z}_N}} Q(\boldsymbol{\rho}_{n-1}, \rho_n|\boldsymbol{\rho}_n)B_{n-1}(\lambda_{n,k})\Psi_{\rho_{n-1}} \otimes \psi^{(n)}_{\rho_n+1}. \tag{80}$$

From (72) we know how to apply B_{n-1} to $\Psi_{\rho_{n-1}}$:

$$\begin{aligned} B_{n-1}(\lambda_{n,k})\Psi_{\rho_{n-1}} &= \lambda_{n,k}r_{n-1,0}\omega^{-\rho_{n-1,0}}\prod_{s=1}^{n-2}(-r_{n,k}\omega^{-\rho_{n,k}} + r_{n-1,s}\omega^{-\rho_{n-1,s}})\Psi_{\rho_{n-1}} \\ &= \lambda_{n,k}\tilde{r}_{n-1}\omega^{-\tilde{\rho}_{n-1}}\left(\prod_{s=1}^{n-2} y^{n,k}_{n-1,s}\frac{w_{p^{n,k}_{n-1,s}}(\rho_{n-1,s}-\rho_{n,k}-1)}{w_{p^{n,k}_{n-1,s}}(\rho_{n-1,s}-\rho_{n,k})}\right)\Psi_{\rho_{n-1}}. \end{aligned} \tag{81}$$

Using (34) we find the action of the inverse of the operator $\mathbf{u}_n^{-1}(a_n - b_n\mathbf{v}_n)$ on $\psi^{(n)}_{\rho_n}$:

$$\left(\mathbf{u}_n^{-1}(a_n - b_n\mathbf{v}_n)\right)^{-1}\psi^{(n)}_{\rho_n} = r_n^{-1}\omega^{\rho_n}\psi^{(n)}_{\rho_n}. \tag{82}$$

Shifting the summation over ρ_n in (80) and then applying (82), we obtain

$$A_n(\lambda_{n,k})\Psi_{\rho_n} = -r_n^{-1}F_n(\lambda_{n,k}/\omega)/\lambda_{n,k} \times \sum_{\substack{\rho_{n-1}\in(\mathbb{Z}_N)^{n-1}\\ \rho_n\in\mathbb{Z}_N}} Q(\boldsymbol{\rho}_{n-1}, \rho_n - 1|\boldsymbol{\rho}_n)\omega^{\rho_n}B_{n-1}(\lambda_{n,k})\Psi_{\rho_{n-1}} \otimes \psi^{(n)}_{\rho_n}. \tag{83}$$

Finally, using (81) and observing that

$$Q(\boldsymbol{\rho}_{n-1}, \rho_n - 1|\boldsymbol{\rho}_n)\omega^{\rho_n-\tilde{\rho}_{n-1}}\prod_{s=1}^{n-2}\frac{w_{p^{n,k}_{n-1,s}}(\rho_{n-1,s}-\rho_{n,k}-1)}{w_{p^{n,k}_{n-1,s}}(\rho_{n-1,s}-\rho_{n,k})} = \omega^{-\tilde{\rho}_n+\rho_{n,0}}Q\left(\boldsymbol{\rho}_{n-1}, \rho_n\middle|\boldsymbol{\rho}_n^{+k}\right)$$

we come to (70).

Formula (71) for $\mathbf{V}_n\Psi_{\rho_n}$.

Using $\mathbf{V}_{n-1}\Psi_{\rho_{n-1}} = \Psi_{\rho^{+0}_{n-1}}$ and $\mathbf{v}_n\psi^{(n)}_{\rho_n} = \psi^{(n)}_{\rho_n+1}$, we have

$$\begin{aligned} \mathbf{V}_{n-1}\mathbf{v}_n\Psi_{\rho_n} &= \sum_{\substack{\rho_{n-1}\in(\mathbb{Z}_N)^{n-1}\\ \rho_n\in\mathbb{Z}_N}} Q(\boldsymbol{\rho}_{n-1}, \rho_n|\boldsymbol{\rho}_n)\Psi_{\rho^{+0}_{n-1}} \otimes \psi^{(n)}_{\rho_n+1} \\ &= \sum_{\substack{\rho_{n-1}\in(\mathbb{Z}_N)^{n-1}\\ \rho_n\in\mathbb{Z}_N}} Q\left(\boldsymbol{\rho}^{-0}_{n-1}, \rho_n - 1\middle|\boldsymbol{\rho}_n\right)\Psi_{\rho_{n-1}} \otimes \psi^{(n)}_{\rho_n}, \end{aligned} \tag{84}$$

where in the second line we have shifted the summation variables $\rho_{n-1,0}$ and ρ_n. Now considering the explicit form (64) for $Q(\boldsymbol{\rho}_{n-1}, \rho_n|\boldsymbol{\rho}_n)$, we read off that

$$Q\left(\boldsymbol{\rho}^{-0}_{n-1}, \rho_n - 1\middle|\boldsymbol{\rho}_n\right) = Q\left(\boldsymbol{\rho}_{n-1}, \rho_n\middle|\boldsymbol{\rho}^{+0}_n\right) \tag{85}$$

which gives (71).

Formula (66) for $A_n(\lambda)\Psi_{\rho_n}$.

The operator $A_n(\lambda)$ is a polynomial in λ of nth order. From (6) and (10) we immediately read off its highest and lowest coefficients:

$$A_n(\lambda) = 1 + \cdots + \lambda^n\varkappa_1\varkappa_2\cdots\varkappa_n\mathbf{V}_n. \tag{86}$$

Using (71) we know how these terms act on Ψ_{ρ_n} and if in addition we use the action of A_n at the $n-1$ particular values given in (70), we can reconstruct the action of the whole polynomial $A_n(\lambda)$ on Ψ_{ρ_n} uniquely. Comparing this to (66) we see that formula (66) is that which satisfies all these data. Therefore, by uniqueness formula (66) is that which we are looking for.

This completes the proof of theorem 1. $\square$

4. Action of D_n on the eigenstates of B_n

In order to obtain the action of $D_n(\lambda)$ on Ψ_{ρ_n}, we use the notion of the quantum determinant $\det_q T_n(\lambda)$ of the monodromy matrix. Since the rank of the matrix $R(\omega\lambda,\lambda)$ is 1, the definition of the quantum determinant is given by

$$R(\omega\lambda,\lambda)T_n^{(1)}(\omega\lambda)T_n^{(2)}(\lambda) = T_n^{(2)}(\lambda)T_n^{(1)}(\omega\lambda)R(\omega\lambda,\lambda) =: \det_q T_n(\lambda)\cdot R(\omega\lambda,\lambda). \tag{87}$$

Explicitly, we have

$$\det_q T_n(\lambda) = A_n(\omega\lambda)D_n(\lambda) - C_n(\omega\lambda)B_n(\lambda). \tag{88}$$

Using (10) and (87), we obtain the factorization property of the quantum determinant

$$\det_q T_n(\lambda) = \det_q L_1(\lambda)\cdot\det_q L_2(\lambda)\cdots\det_q L_n(\lambda).$$

For a single L-operator, using (68), (88) gives $\det_q L_m(\lambda) = \mathbf{v}_m F_m(\lambda)$. So

$$A_n(\omega\lambda)D_n(\lambda) - C_n(\omega\lambda)B_n(\lambda) = \mathbf{V}_n\cdot\prod_{m=1}^{n} F_m(\lambda). \tag{89}$$

Acting by both sides of this identity on Ψ_{ρ_n}, fixing $\lambda = \lambda_{n,k}$ (i.e. at the zeros of the eigenvalue polynomial of $B_n(\lambda)$) and using the inverse of (70) with (67), we see that, very similar to $A_n(\lambda_{n,k})$, also $D_n(\lambda_{n,k})$ acts as a shift operator on Ψ_{ρ_n}, compare (27):

$$D_n(\lambda_{n,k})\Psi_{\rho_n} = \tilde{\varphi}_k(\rho_n')\Psi_{\rho_n^{+0,-k}}, \qquad \tilde{\varphi}_k(\rho_n') = -\frac{r_n}{\tilde{r}_{n-1}}\frac{\omega^{\tilde{\rho}_n-\rho_{n,0}-1}}{\prod_{s=1}^{n-2} y_{n-1,s}^{n,k}}\prod_{m=1}^{n-1}F_m(\lambda_{n,k}). \tag{90}$$

Note that $D_n(\lambda_{n,k})$ shifts $\rho_{n,k}$ in the opposite direction as $A_n(\lambda_{n,k})$ (see (67) and (70)) and $D_n(\lambda_{n,k})$ also shifts $\rho_{n,0}$. The shift in $\rho_{n,0}$ is due to the operator $\mathbf{V}_n$ on the right-hand side of (89). Apart from the shifts just mentioned, applying the inverse of $A_n(\omega\lambda)$ has cancelled in (90) the last factor $m = n$ of the quantum determinant (89). Analogously to (26), using the particular values (90) and reading off the coefficients of λ^0 and λ^n directly from (10), we obtain the following interpolation formula for $D_n(\lambda)\Psi_{\rho_n}$:

$$\begin{aligned} D_n(\lambda)\Psi_{\rho_n} &= \sum_{k=1}^{n-1}\left(\prod_{s\neq k}\frac{\lambda-\lambda_{n,s}}{\lambda_{n,k}-\lambda_{n,s}}\right)\frac{\lambda}{\lambda_{n,k}}\tilde{\varphi}_k(\rho_n')\Psi_{\rho_n^{+0,-k}} \\ &\quad + \prod_{s=1}^{n-1}\left(1-\frac{\lambda}{\lambda_{n,s}}\right)\prod_{m=1}^{n}\frac{b_m d_m}{\varkappa_m}\Psi_{\rho_n^{+0}} + \lambda\prod_{m=1}^{n}a_m c_m\prod_{s=1}^{n-1}(\lambda-\lambda_{n,s})\Psi_{\rho_n}. \end{aligned} \tag{91}$$

5. Periodic model: Baxter equation and functional relations

5.1. The Baxter equations

After having determined the eigenvalues and eigenvectors of the auxiliary system, we now perform the first step of the programme exposed in subsection 1.3, i.e. the calculation of the eigenvalues and eigenvectors of the inhomogeneous n-site *periodic* BBS chain model with the transfer matrix (11), (14). Following the ideas of the papers [8, 9, 18–20], we are looking for eigenvectors of $\mathbf{t}_n(\lambda)$ as linear combinations of the eigenvectors Ψ_{ρ_n} of the auxiliary system.

It is convenient to go by Fourier transform in $\rho_{n,0}$ from Ψ_{ρ_n} to a basis of eigenvectors of $\mathbf{V}_n$ (and therefore of the Hamiltonians $\mathbf{H}_0$ and $\mathbf{H}_n$, see (14), (15))

$$\tilde{\Psi}_{\rho,\rho'_n} = \sum_{\rho_{n,0}\in\mathbb{Z}_N} \omega^{-\rho\cdot\rho_{n,0}}\Psi_{\rho_n}, \qquad \mathbf{V}_n\tilde{\Psi}_{\rho,\rho'_n} = \omega^{\rho}\tilde{\Psi}_{\rho,\rho'_n}. \tag{92}$$

A shift of $\rho_{n,0}$ in Ψ_{ρ_n} is replaced by a multiplication of $\tilde{\Psi}_{\rho,\rho'_n}$ by powers of ω. So from (66) and (91), the action of $\mathbf{t}_n(\lambda)$ on $\tilde{\Psi}_{\rho,\rho'_n}$ becomes

$$\begin{aligned}\mathbf{t}_n(\lambda)\tilde{\Psi}_{\rho,\rho'_n} &= \sum_{k=1}^{n-1}\left(\prod_{s\neq k}\frac{\lambda-\lambda_{n,s}}{\lambda_{n,k}-\lambda_{n,s}}\right)\frac{\lambda}{\lambda_{n,k}}\left(\varphi_k(\boldsymbol{\rho}'_n)\tilde{\Psi}_{\rho,\boldsymbol{\rho}'^{+k}_n}+\omega^{\rho}\tilde{\varphi}_k(\boldsymbol{\rho}'_n)\tilde{\Psi}_{\rho,\boldsymbol{\rho}'^{-k}_n}\right)\\ &\quad+\left\{\left(1+\omega^{\rho}\prod_{m=1}^{n}\frac{b_m d_m}{\varkappa_m}\right)\prod_{s=1}^{n-1}\left(1-\frac{\lambda}{\lambda_{n,s}}\right)\right.\\ &\quad\left.+\lambda\left(\omega^{\rho}\prod_{m=1}^{n}\varkappa_m+\prod_{m=1}^{n}a_m c_m\right)\prod_{s=1}^{n-1}(\lambda-\lambda_{n,s})\right\}\tilde{\Psi}_{\rho,\rho'_n},\end{aligned} \tag{93}$$

where we have taken into account that $\varphi_k(\boldsymbol{\rho}'_n)$ and $\tilde{\varphi}_k(\boldsymbol{\rho}'_n)$ are independent of $\rho_{n,0}$. Of course, since $\mathbf{t}_n(\lambda)$ commutes with $\mathbf{V}_n$, in (93) there is no coupling between sectors of different ρ and we get separate equations for the different 'charge' quantum numbers ρ which often will not be indicated explicitly.

Let $\Phi_{\rho,\mathbf{E}}$ be an eigenvector of $\mathbf{t}_n(\lambda)$ with the eigenvalue

$$t_n(\lambda|\rho,\mathbf{E}) = E_0 + E_1\lambda + \cdots + E_{n-1}\lambda^{n-1} + E_n\lambda^n, \tag{94}$$

where $\mathbf{E} = \{E_1,\ldots,E_{n-1}\}$ and from (15) the values of E_0 and E_n are

$$E_0 = 1+\omega^{\rho}\prod_{m=1}^{n}\frac{b_m d_m}{\varkappa_m}, \qquad E_n = \prod_{m=1}^{n}a_m c_m + \omega^{\rho}\prod_{m=1}^{n}\varkappa_m. \tag{95}$$

We are looking for $\Phi_{\rho,\mathbf{E}}$ to be of the form

$$\Phi_{\rho,\mathbf{E}} = \sum_{\rho'_n} Q(\boldsymbol{\rho}'_n|\rho,\mathbf{E})\tilde{\Psi}_{\rho,\rho'_n}. \tag{96}$$

From (93) we get a difference equation for $Q(\boldsymbol{\rho}'_n|\rho,\mathbf{E})$ with respect to variables $\boldsymbol{\rho}'_n$ which depends on λ:

$$\begin{aligned}t_n(\lambda|\rho,\mathbf{E})Q(\boldsymbol{\rho}'_n|\rho,\mathbf{E}) &= \sum_{k=1}^{n-1}\frac{\lambda}{\omega\lambda_{n,k}}\varphi_k\left(\boldsymbol{\rho}'^{-k}_n\right)Q\left(\boldsymbol{\rho}'^{-k}_n|\rho,\mathbf{E}\right)\prod_{s\neq k}\frac{\lambda-\lambda_{n,s}}{\omega\lambda_{n,k}-\lambda_{n,s}}\\ &\quad+\sum_{k=1}^{n-1}\frac{\omega\lambda}{\lambda_{n,k}}\omega^{\rho}\tilde{\varphi}_k\left(\boldsymbol{\rho}'^{+k}_n\right)Q\left(\boldsymbol{\rho}'^{+k}_n|\rho,\mathbf{E}\right)\prod_{s\neq k}\frac{\lambda-\lambda_{n,s}}{\omega^{-1}\lambda_{n,k}-\lambda_{n,s}}\\ &\quad+\left\{\left(1+\omega^{\rho}\prod_{m=1}^{n}\frac{b_m d_m}{\varkappa_m}\right)\prod_{s=1}^{n-1}\left(1-\frac{\lambda}{\lambda_{n,s}}\right)\right.\\ &\quad\left.+\lambda\left(\omega^{\rho}\prod_{m=1}^{n}\varkappa_m+\prod_{m=1}^{n}a_m c_m\right)\prod_{s=1}^{n-1}(\lambda-\lambda_{n,s})\right\}Q(\boldsymbol{\rho}'_n|\rho,\mathbf{E}).\end{aligned} \tag{97}$$

Substituting sequentially $\lambda = \lambda_{n,k}$, $k = 1, 2, \ldots, n-1$, we obtain a system of difference equations with respect to the variables $\boldsymbol{\rho}'_n$:

$$t_n(\lambda_{n,k}|\rho,\mathbf{E})\,Q(\boldsymbol{\rho}'_n|\rho,\mathbf{E}) = \left(\prod_{s\neq k}\frac{\lambda_{n,k}-\lambda_{n,s}}{\omega\lambda_{n,k}-\lambda_{n,s}}\right)\omega^{-1}\varphi_k\big(\boldsymbol{\rho}'^{-k}_n\big)\,Q\big(\boldsymbol{\rho}'^{-k}_n|\rho,\mathbf{E}\big)$$
$$+\left(\prod_{s\neq k}\frac{\lambda_{n,k}-\lambda_{n,s}}{\omega^{-1}\lambda_{n,k}-\lambda_{n,s}}\right)\omega^{\rho+1}\tilde{\varphi}_k\big(\boldsymbol{\rho}'^{+k}_n\big)\,Q\big(\boldsymbol{\rho}'^{+k}_n|\rho,\mathbf{E}\big),\qquad k=1,\ldots,n-1. \tag{98}$$

In analogy to [9, 18, 19], we can decouple these difference equations using a Sklyanin's measure, namely, by introducing $\widetilde{Q}(\boldsymbol{\rho}'_n|\rho,\mathbf{E})$ defined as

$$Q(\boldsymbol{\rho}'_n|\rho,\mathbf{E}) = \frac{\widetilde{Q}(\boldsymbol{\rho}'_n|\rho,\mathbf{E})}{\prod_{\substack{s,s'=1\\(s\neq s')}}^{n-1} w_{p^{n,s'}_{n,s}}(\rho_{n,s}-\rho_{n,s'})}. \tag{99}$$

Rewriting (98) in terms of $\widetilde{Q}$ produces factors $R_\pm$ in both terms of the right-hand side:

$$t_n(\lambda_{n,k}|\rho,\mathbf{E})\,\widetilde{Q}(\boldsymbol{\rho}'_n|\rho,\mathbf{E}) = \left(\prod_{s\neq k}\frac{\lambda_{n,k}-\lambda_{n,s}}{\omega\lambda_{n,k}-\lambda_{n,s}}\right)\omega^{-1}\varphi_k\big(\boldsymbol{\rho}'^{-k}_n\big)R_-\widetilde{Q}\big(\boldsymbol{\rho}'^{-k}_n|\rho,\mathbf{E}\big)$$
$$+\left(\prod_{s\neq k}\frac{\lambda_{n,k}-\lambda_{n,s}}{\omega^{-1}\lambda_{n,k}-\lambda_{n,s}}\right)\omega^{\rho+1}\tilde{\varphi}_k\big(\boldsymbol{\rho}'^{+k}_n\big)R_+\widetilde{Q}\big(\boldsymbol{\rho}'^{+k}_n|\rho,\mathbf{E}\big),\qquad k=1,\ldots,n-1, \tag{100}$$

where

$$R_+ = \prod_{\substack{s=1\\s\neq k}}^{n-1}\frac{w_{p^{n,k}_{n,s}}(\rho_{n,s}-\rho_{n,k})}{w_{p^{n,k}_{n,s}}(\rho_{n,s}-\rho_{n,k}-1)}\,\frac{w_{p^{n,s}_{n,k}}(\rho_{n,k}-\rho_{n,s})}{w_{p^{n,s}_{n,k}}(\rho_{n,k}-\rho_{n,s}+1)}$$
$$= \prod_{\substack{s=1\\s\neq k}}^{n-1}\frac{y^{n,k}_{n,s}}{1-x^{n,k}_{n,s}\omega^{\rho_{n,s}-\rho_{n,k}}}\,\frac{1-x^{n,s}_{n,k}\omega^{\rho_{n,k}-\rho_{n,s}+1}}{y^{n,s}_{n,k}} = \prod_{\substack{s=1\\s\neq k}}^{n-1}\frac{y^{n,k}_{n,s}}{y^{n,s}_{n,k}}\,\frac{\lambda_{n,s}}{\lambda_{n,k}}\,\frac{\omega\lambda_{n,s}-\lambda_{n,k}}{\lambda_{n,k}-\lambda_{n,s}},$$

and analogously R_-. We see that passing from Q to $\widetilde{Q}$ the brackets containing differences of terms $\lambda_{n,l}$ in (100) are cancelled and so the equations decouple. This means that in terms of $\widetilde{Q}$, the difference equations (98) admit the *separation of variables*:

$$\widetilde{Q}(\boldsymbol{\rho}'_n|\mathbf{E}) = \prod_{k=1}^{n-1}\tilde{q}_k(\rho_{n,k}). \tag{101}$$

Inserting the explicit expressions for $\varphi_k\big(\boldsymbol{\rho}'^{-k}_n\big)$ and $\tilde{\varphi}_k\big(\boldsymbol{\rho}'^{+k}_n\big)$, we obtain Baxter-type difference equations for the functions $\tilde{q}_k(\rho_{n,k})$:

$$t_n(\lambda_{n,k}|\rho,\mathbf{E})\tilde{q}_k(\rho_{n,k}) = \Delta_+(\lambda_{n,k})\tilde{q}_k(\rho_{n,k}+1) + \Delta_-(\omega\lambda_{n,k})\tilde{q}_k(\rho_{n,k}-1) \tag{102}$$

with

$$\Delta_+(\lambda) = (\omega^\rho/\chi_k)(\lambda/\omega)^{1-n}\prod_{m=1}^{n-1}F_m(\lambda/\omega),\qquad \Delta_-(\lambda) = \chi_k(\lambda/\omega)^{n-1}F_n(\lambda/\omega), \tag{103}$$

where $\Delta_\pm(\lambda)$ depends on k through χ_k:

$$\chi_k = \frac{r_{n,0}\tilde{r}_{n-1}}{r_n\tilde{r}_n}\left(\prod_{\substack{s=1\\ s\neq k}}^{n-1} y_{n,k}^{n,s}\big/ y_{n,s}^{n,k}\right)\prod_{s=1}^{n-2} y_{n-1,s}^{n,k}. \tag{104}$$

In what follows, we will mainly use $t(\lambda)$ instead of $t_n(\lambda|\rho,\mathbf{E})$. In fact, the system of linear homogeneous equations (102) with respect to $\tilde{q}_k(\rho_{n,k})$, $\rho_{n,k}\in\mathbb{Z}_N$, is not completely defined. Since $E_1, E_2, \ldots, E_{n-1}$ are unknown, the coefficients $t(\lambda_{n,k})$ are also unknown. The requirement on the system of homogeneous equations (102) for some fixed $k, k = 1, 2, \ldots, n-1$, to have a nontrivial solution leads to the requirement that the matrix of coefficients must be degenerate. The latter gives a relation for the values $E_0, E_1, \ldots, E_n$ entering $t(\lambda)$. Taking all such relations corresponding to all $k = 1, 2, \ldots, n-1$ and using the values of E_0 and E_n given in (95), at least in principle we can find the possible values of $E_1, \ldots, E_{n-1}$. This fixes $t(\lambda)$. Then for every $k, k = 1, 2, \ldots, n-1$, we solve (102) to find $\tilde{q}_k(\rho_{n,k})$ for $\rho_{n,k}\in\mathbb{Z}_N$. (These difference equations have three terms and cannot be solved in terms of the functions w_p.) This gives us finally $Q(\rho'_n|\rho,\mathbf{E})$ and therefore the eigenvectors of the periodic BBS model:

$$\Phi_{\rho,\mathbf{E}} = \sum_{\rho_n=(\rho_{n,0},\rho'_n)} \omega^{-\rho\cdot\rho_{n,0}} Q(\rho'_n|\rho,\mathbf{E})\Psi_{\rho_n}.$$

5.2. Role of the functional relations

Now we will show that the mentioned requirement on the systems of homogeneous equations (102) for all k to have a nontrivial solution is equivalent to functional relations [1, 3, 4] of the $\tau^{(2)}$ model. We define $\tau^{(0)}(\lambda) = 0$, $\tau^{(1)}(\lambda) = 1$, $\tau^{(2)}(\lambda) = t(\lambda)$ (see (94), (95)) and

$$\tau^{(j+1)}(\lambda) = \tau^{(2)}(\omega^{j-1}\lambda)\tau^{(j)}(\lambda) - \omega^{\rho} z(\omega^{j-1}\lambda)\tau^{(j-1)}(\lambda), \qquad j = 2, 3, \ldots, N, \tag{105}$$

where

$$z(\lambda) = \omega^{-\rho}\Delta_+(\lambda)\Delta_-(\lambda) = \prod_{m=1}^{n} F_m(\lambda/\omega). \tag{106}$$

The appearance of the monodromy determinant (68) in the fusion relation is a direct consequence of the fusion procedure (see [22, 23]).

The fusion hierarchy can be used to find eigenvalues of the transfer matrices in lattice integrable models. A key tool here is, in addition to (105), to demand a 'truncation' identity which allows us to express $\tau^{(j)}(\lambda)$ for some value j through $\tau^{(i)}(\lambda)$ with $i < j$. A combination of the fusion hierarchy and truncation identity allows one to obtain an equation for $\tau^{(2)}(\lambda) = t(\lambda)$. This method was applied to many integrable models, in particular, to the RSOS models in [24] and to the root-of-unity lattice vertex models in [25]. The functional relations for the $\tau^{(2)}$ model for $N = 3$ and the superintegrable case were first guessed in [12] and have been solved to some extent in [26].

The goal of the present section is to prove that the relations to determine the values $E_1, \ldots, E_{n-1}$ entering $t(\lambda)$ also have the form of a truncation identity. We formulate this statement as follows.

Theorem 2. *The polynomial $\tau^{(N+1)}(\lambda)$ satisfies the 'truncation' identity*

$$\tau^{(N+1)}(\lambda) - \omega^{\rho} z(\lambda)\tau^{(N-1)}(\omega\lambda) = \mathcal{A}_n(\lambda^N) + \mathcal{D}_n(\lambda^N) \tag{107}$$

if and only if the system of homogeneous equations (102) for all k has a nontrivial solution.

Note that the classical polynomial $\mathcal{A}_n(\lambda^N) + \mathcal{D}_n(\lambda^N)$ corresponds to $\alpha_q + \bar{\alpha}_q$ in [1].

Proof. Let $t(\lambda)$ be a polynomial (94), (95) such that the systems of homogeneous equations (102) for all k have a nontrivial solution. We shall show that the polynomial $P(\lambda) = \tau^{(N+1)}(\lambda) - \omega^\rho z(\lambda)\tau^{(N-1)}(\omega\lambda)$ on the left-hand side of (107) is equal to $\mathcal{A}_n(\lambda^N) + \mathcal{D}_n(\lambda^N)$. With this aim we introduce the matrix

$$\Gamma(\lambda) = \begin{pmatrix} \tau^{(2)}(\lambda) & \omega^\rho z(\lambda) \\ -1 & 0 \end{pmatrix}.$$

Then it is easy to verify from (105) by induction that

$$\Gamma(\omega^{j-1}\lambda)\cdots\Gamma(\omega\lambda)\Gamma(\lambda) = \begin{pmatrix} \tau^{(j+1)}(\lambda) & \omega^\rho z(\lambda)\tau^{(j)}(\omega\lambda) \\ -\tau^{(j)}(\lambda) & -\omega^\rho z(\lambda)\tau^{(j-1)}(\omega\lambda) \end{pmatrix}$$

and we see that

$$P(\lambda) = \operatorname{tr}\Gamma(\omega^{N-1}\lambda)\cdots\Gamma(\omega\lambda)\Gamma(\lambda). \tag{108}$$

This relation shows the invariance of $P(\lambda)$ under cyclic shifting $\lambda \to \omega\lambda$. It means that in fact $P(\lambda)$ depends only on λ^N. We denote $\mathcal{P}(\lambda^N) = P(\lambda)$. Thus, we have to show that $\mathcal{P}(\lambda^N) = \mathcal{A}_n(\lambda^N) + \mathcal{D}_n(\lambda^N)$. The direct calculation gives that the coefficients of λ^0 and λ^{Nn} on both sides of this equation coincide. In order to calculate the coefficient in front of λ^0 in the trace of the product of Γ-matrices (108), one has to substitute

$$\Gamma(\lambda) \to \begin{pmatrix} 1 + \omega^\rho \prod_{m=1}^n b_m d_m/\varkappa_m & \omega^\rho \prod_{m=1}^n b_m d_m/\varkappa_m \\ -1 & 0 \end{pmatrix}, \tag{109}$$

where only the lowest terms in λ in the matrix elements were kept. Therefore, the lowest term in λ in $P(\lambda)$ is

$$\operatorname{tr}\begin{pmatrix} 1 + \omega^\rho \prod_{m=1}^n \frac{b_m d_m}{\varkappa_m} & \omega^\rho \prod_{m=1}^n \frac{b_m d_m}{\varkappa_m} \\ -1 & 0 \end{pmatrix}^N.$$

Finding the eigenvalues of the matrix (109), one can easily calculate the latter trace which is the lowest term $\mathcal{P}(0)$ and identify it with the lowest term

$$\mathcal{A}_n(0) + \mathcal{D}_n(0) = 1 + \prod_{m=1}^n \frac{b_m^N d_m^N}{\varkappa_m^N}$$

of the polynomial $\mathcal{A}_n(\lambda^N) + \mathcal{D}_n(\lambda^N)$ calculated by means of relations (45) and (46). The case of the coefficients in front of λ^{Nn} can be treated analogously.

Therefore, to prove theorem 2 it remains to prove that

$$P(\lambda_{n,k}) = \mathcal{A}_n(\lambda_{n,k}^N) + \mathcal{D}_n(\lambda_{n,k}^N), \qquad k = 1, 2, \ldots, n-1, \tag{110}$$

where $\lambda_{n,k}$ are given by (31) and $\lambda_{n,k}^N = \epsilon r_{n,k}^N$ are zeros of the polynomial (47). Let us fix some k and $\rho_{n,k}$ and denote the matrix of the coefficients of (102) with respect to the variables $\tilde{q}_k(\rho_{n,k}), \tilde{q}_k(\rho_{n,k}-1), \ldots, \tilde{q}_k(\rho_{n,k}-N+1)$ by $\mathcal{M}$:

$$\mathcal{M} = \begin{pmatrix} t_0 & -\Delta_1^- & 0 & \ldots & 0 & -\Delta_0^+ \\ -\Delta_1^+ & t_1 & -\Delta_2^- & \ldots & 0 & 0 \\ 0 & -\Delta_2^+ & t_2 & \ldots & 0 & 0 \\ & & \ldots & \ldots & \ldots & \\ -\Delta_0^- & 0 & 0 & \ldots & -\Delta_{N-1}^+ & t_{N-1} \end{pmatrix}, \tag{111}$$

where we abbreviated $t_j = t(\omega^j\lambda_{n,k})$, $\Delta_j^\pm = \Delta_\pm(\omega^j\lambda_{n,k})$. In order (102) to have a nontrivial solution, the matrix $\mathcal{M}$ must be degenerate. Let $\mathcal{M}'$ be the matrix which has the same matrix elements as $\mathcal{M}$ except for $\mathcal{M}'_{1,N} = 0$ and $\mathcal{M}'_{N,1} = 0$. Then, using the recursive definition (105) of $\tau^{(j)}(\lambda)$ and (106), it is easy to show that the principal minor corresponding to the first j, $j \leqslant N$ rows of the matrix $\mathcal{M}'$ gives $\tau^{(j+1)}(\lambda_{n,k})$. Calculating the determinant of the matrix (111) and equating it to zero, we obtain

$$\det \mathcal{M} = \tau^{(N+1)}(\lambda_{n,k}) - \omega^{\rho} z(\lambda_{n,k})\tau^{(N-1)}(\omega\lambda_{n,k}) - \prod_{s\in\mathbb{Z}_N} \Delta_+(\lambda_{n,k}\omega^s) - \prod_{s\in\mathbb{Z}_N} \Delta_-(\lambda_{n,k}\omega^s) = 0. \tag{112}$$

Further,

$$\prod_{s\in\mathbb{Z}_N} \Delta_-(\lambda_{n,k}\omega^s) = \chi_k^N (-1)^{n-1} r_{n,k}^{N(n-1)} \det \mathcal{L}_n\left(\lambda_{n,k}^N\right) = \epsilon \frac{\mathcal{B}_{n-1}\left(\lambda_{n,k}^N\right)}{r_n^N \lambda_{n,k}^N} \det \mathcal{L}_n\left(\lambda_{n,k}^N\right),$$

where we have used (103), (50), (104), (54), (47) and

$$\chi_k^N = \frac{(-1)^n r_{n-1,0}^N}{r_n^N \left(r_{n,k}^N\right)^{n-1}} \prod_{s=1}^{n-2} \left(r_{n-1,s}^N - r_{n,k}^N\right) = \frac{(-1)^{n-1}\mathcal{B}_{n-1}\left(\lambda_{n,k}^N\right)}{r_n^N \left(r_{n,k}^N\right)^n}.$$

Evaluating (49) at $m = n$ and $\lambda = \lambda_{n,k}$ so that $\mathcal{B}_n\left(\lambda_{n,k}^N\right) = 0$, finally we obtain

$$\prod_{s\in\mathbb{Z}_N} \Delta_-(\lambda_{n,k}\omega^s) = \mathcal{A}_n\left(\lambda_{n,k}^N\right).$$

Substituting $\lambda = \lambda_{n,k}$ into

$$\begin{aligned} \det \mathcal{T}_n(\lambda^N) &= \mathcal{A}_n(\lambda^N)\mathcal{D}_n(\lambda^N) - \mathcal{B}_n(\lambda^N)\mathcal{C}_n(\lambda^N) = \prod_{m=1}^{n} \det \mathcal{L}_m(\lambda^N) \\ &= \prod_{m=1}^{n}\prod_{s\in\mathbb{Z}_N} F_m(\lambda\omega^{s-1}) = \prod_{s\in\mathbb{Z}_N} z(\lambda\omega^s) = \prod_{s\in\mathbb{Z}_N} (\Delta_+(\lambda\omega^s)\cdot\Delta_-(\lambda\omega^s)), \end{aligned} \tag{113}$$

we get

$$\prod_{s\in\mathbb{Z}_N} \Delta_+(\lambda_{n,k}\omega^s) = \mathcal{D}_n\left(\lambda_{n,k}^N\right).$$

Using (112) we obtain (110).

Conversely, if we have the polynomials $\tau^{(N-1)}(\lambda)$ and $\tau^{(N+1)}(\lambda)$ built from $\tau^{(2)}(\lambda) = t(\lambda)$ (see (94), (95)) by the recursion (105) and satisfying (107), we get (112) at particular values of λ. This means that the Baxter equations (102) have nontrivial solutions.

This completes the proof of theorem 2. □

6. Periodic homogeneous BBS model for N = 2

6.1. Solution of the Baxter equations

In this section, we consider in more detail the case of the $N = 2$ periodic homogeneous BBS model, where $\omega = -1$. By homogeneous we mean that the parameters a, b, c, d and $\varkappa$ are taken to be the same for all sites. As was shown in [16], for $N = 2$ and with arbitrary homogeneous parameters this model is a particular case ('free fermion point') of the generalized Ising model.

We will find the eigenvalues $t_n(\lambda|\rho,\mathbf{E})$ of the transfer matrix $\mathbf{t}_n(\lambda)$ using a functional relation (see also [30], where a similar calculation is presented). We use the short notation $t(\lambda)$ for $t_n(\lambda|\rho,\mathbf{E})$. From the previous section, we have

$$t(\lambda) = 1 + (-1)^\rho \frac{b^n d^n}{\varkappa^n} + E_1\lambda + \cdots + E_{n-1}\lambda^{n-1} + \lambda^n(a^n c^n + (-1)^\rho \varkappa^n). \tag{114}$$

Using (105) for $j = 2$ and eliminating $\tau^{(3)}$ by (107), we get the functional relation

$$t(\lambda)t(-\lambda) = (-1)^\rho (z(\lambda) + z(-\lambda)) + \mathcal{A}_n(\lambda^2) + \mathcal{D}_n(\lambda^2), \tag{115}$$

which we shall use to find $t(\lambda)$. Equivalently, we could have obtained (115) by multiplying together the two Baxter equations (102) for $\lambda_{n,k} = \pm r_{n,k}$.

Postponing for a moment the derivation (which will be supplied after (126)), let us anticipate that (115) can be rewritten as

$$t(\lambda)t(-\lambda) = (-1)^n \prod_q (A(q)\lambda^2 - C(q) + 2\mathrm{i}B(q)\lambda), \tag{116}$$

where

$$\begin{aligned} &A(q) = a^2c^2 - 2\varkappa ac\cos q + \varkappa^2, \qquad B(q) = (ad - bc)\sin q, \\ &C(q) = 1 - 2\frac{bd}{\varkappa}\cos q + \frac{b^2d^2}{\varkappa^2}, \end{aligned} \tag{117}$$

q is running over the set $\pi(2s + 1 - \rho)/n$, $s = 0, 1, \ldots, n-1$. Factorizing (116) we get

$$t(\lambda)t(-\lambda) = (-1)^n \prod_q A(q)(\lambda - \lambda_q)(\lambda + \lambda_{-q}) \tag{118}$$

with

$$\lambda_q = \frac{1}{A(q)}(\sqrt{D(q)} - \mathrm{i}B(q)), \qquad D(q) = A(q)C(q) - B(q)^2. \tag{119}$$

We fix the sign of $\sqrt{D(q)}$ by the conditions

$$\sqrt{D(q)} = \sqrt{D(-q)}, \qquad q \neq 0, \pi, \tag{120}$$

$$\sqrt{D(0)} = (\varkappa - ac)(1 - bd/\varkappa), \qquad \sqrt{D(\pi)} = (\varkappa + ac)(1 + bd/\varkappa). \tag{121}$$

In what follows, we shall need the relations

$$\prod_q A(q) = \prod_q (\varkappa - \mathrm{e}^{\mathrm{i}q}ac)(\varkappa - \mathrm{e}^{-\mathrm{i}q}ac) = (a^n c^n + (-1)^\rho \varkappa^n)^2, \tag{122}$$

$$\prod_q (\sqrt{D(q)} - \mathrm{i}B(q)) = \prod_q (\varkappa - \mathrm{e}^{\mathrm{i}q}ac)(1 - \mathrm{e}^{\mathrm{i}q}bd/\varkappa). \tag{123}$$

The last relation can be obtained by grouping terms with opposite signs of q (modulo 2π) and using the definition of $\sqrt{D(q)}$. Using (122) we get

$$\begin{aligned} t(\lambda)t(-\lambda) &= (-1)^n(a^nc^n + (-1)^\rho\varkappa^n)^2 \prod_q (\lambda - \lambda_q) \cdot \prod_q (\lambda + \lambda_{-q}) \\ &= (-1)^n(a^nc^n + (-1)^\rho\varkappa^n)^2 \prod_q \left(\lambda^2 - \lambda_q^2\right), \end{aligned} \tag{124}$$

where we made the change $q \to -q$ in the second product. From (114) it follows that $t(\lambda)$ can be presented as

$$t(\lambda) = (a^nc^n + (-1)^\rho\varkappa^n) \prod_{s=1}^n (\lambda - \Lambda_s)$$

with zeros Λ_s. Therefore,

$$t(\lambda)t(-\lambda) = (-1)^n(a^nc^n + (-1)^\rho\varkappa^n)^2\prod_{s=1}^{n}\left(\lambda^2 - \Lambda_s^2\right).$$

Comparing with (124), we obtain

$$t(\lambda) = (a^nc^n + (-1)^\rho\varkappa^n)\prod_q(\lambda \pm \lambda_q), \tag{125}$$

where the signs are not yet fixed. To fix these signs, we compare the λ-independent term in (114) with the corresponding term in (125). The latter can be found using

$$\begin{aligned}(a^nc^n + (-1)^\rho\varkappa^n)\prod_q\lambda_q &= (a^nc^n + (-1)^\rho\varkappa^n)^{-1}\prod_q(\sqrt{D(q)} - \mathrm{i}B(q))\\ &= (a^nc^n + (-1)^\rho\varkappa^n)^{-1}\prod_q(\varkappa - \mathrm{e}^{\mathrm{i}q}ac)(1 - \mathrm{e}^{\mathrm{i}q}bd/\varkappa) = (-1)^\rho + b^nd^n/\varkappa^n,\end{aligned} \tag{126}$$

where we took into account (123). Therefore, the number of minus signs in (125) must be even for the sector $\rho = 0$ and odd for $\rho = 1$. Thus, we have obtained 2^n eigenvalues (2^{n-1} each for both $\rho = 0$ and $\rho = 1$). These eigenvalues provide the existence of nontrivial solutions of the system (102) of homogeneous equations. These solutions give the eigenvectors (96), a basis in the space of states of the periodic BBS model for $N = 2$.

We conclude this section supplying the derivation of (116) from (115): using

$$\delta_+(\lambda) := (b + a\varkappa\lambda)(d - c\varkappa\lambda)/\varkappa, \qquad \delta_-(\lambda) := \delta_+(-\lambda) = (b - a\varkappa\lambda)(d + c\varkappa\lambda)/\varkappa,$$

we easily verify the relations $\delta_+^n(\lambda) = z(\lambda)$, $\delta_-^n(\lambda) = z(-\lambda)$, $\delta_+(\lambda)\delta_-(\lambda) = \delta(\lambda^2)$, where $z(\lambda)$ and $\delta(\lambda^2)$ are given by (106) and (A.5), respectively. Taking into account $\mathcal{A}_n(\lambda^2) + \mathcal{D}_n(\lambda^2) = \mathrm{tr}(\mathcal{L}(\lambda^2))^n = x_+^n(\lambda^2) + x_-^n(\lambda^2)$ and the relations

$$\tau(\lambda^2) = \mathrm{tr}\,\mathcal{L}(\lambda^2) = x_+(\lambda^2) + x_-(\lambda^2), \qquad \delta(\lambda^2) = \det\mathcal{L}(\lambda^2) = x_+(\lambda^2)x_-(\lambda^2),$$

we can rewrite the functional relation (115) as

$$\begin{aligned}t(\lambda)t(-\lambda) &= (-1)^\rho(z(\lambda) + z(-\lambda)) + \mathcal{A}_n(\lambda^2) + \mathcal{D}_n(\lambda^2)\\ &= (-1)^\rho\left(\delta_+^n + \delta_-^n\right) + x_+^n + x_-^n\\ &= (-1)^\rho\left(x_+^n + (-1)^\rho\delta_+^n\right)((x_-/\delta_+)^n + (-1)^\rho)\\ &= (-1)^\rho\prod_q(x_+ - \mathrm{e}^{\mathrm{i}q}\delta_+)(x_-/\delta_+ - \mathrm{e}^{\mathrm{i}q})\\ &= (-1)^n\prod_q(\mathrm{e}^{\mathrm{i}q}\delta_+ - \tau(\lambda^2) + \mathrm{e}^{-\mathrm{i}q}\delta_-)\\ &= (-1)^n\prod_q\left\{\left((a^2c^2 + \varkappa^2)\lambda^2 - \frac{b^2d^2}{\varkappa^2} - 1\right)\right.\\ &\quad\left. + 2\left(\frac{bd}{\varkappa} - \lambda^2\varkappa ac\right)\cos q - 2\mathrm{i}\lambda(ad - bc)\sin q\right\},\end{aligned} \tag{127}$$

which confirms (116).

6.2. Relation to the standard Ising model notations

In the homogeneous $N = 2$ case, we have $\omega = -1$ and $\mathbf{u}_k^{-1} = \mathbf{u}_k$, so the cyclic L-operator (6) reduces to

$$L_k(\lambda) = \begin{pmatrix} 1 + \lambda\varkappa\mathbf{v}_k & \lambda\mathbf{u}_k(a - b\mathbf{v}_k) \\ \mathbf{u}_k(c - d\mathbf{v}_k) & \lambda ac + \mathbf{v}_k bd/\varkappa \end{pmatrix}. \tag{128}$$

Let us make the special choice of the parameters $d = bc/a$ and $\lambda = b/(a\varkappa)$. Then

$$L_k(\lambda) = (1 + \mathbf{v}_k b/a)\begin{pmatrix} 1 & \mathbf{u}_k b/\varkappa \\ c\mathbf{u}_k & bc/\varkappa \end{pmatrix} = (1 + \mathbf{v}_k b/a)\begin{pmatrix} 1 \\ c\mathbf{u}_k \end{pmatrix}(1 \quad \mathbf{u}_k b/\varkappa) \tag{129}$$

and the transfer matrix is

$$\mathbf{t}_n(\lambda) = \mathrm{tr}\, L_1(\lambda)L_2(\lambda)\cdots L_n(\lambda) = \prod_{k=1}^{n}(1 + \mathbf{v}_k \cdot b/a) \cdot \prod_{k=1}^{n}(1 + \mathbf{u}_{k-1}\mathbf{u}_k \cdot bc/\varkappa). \tag{130}$$

Recall that due to the periodic boundary conditions $\mathbf{u}_{n+k} \equiv \mathbf{u}_k$. Using

$$\exp(K_1\mathbf{u}_{k-1}\mathbf{u}_k) = \cosh K_1(1+\mathbf{u}_{k-1}\mathbf{u}_k \tanh K_1), \qquad \exp(K_1^*\mathbf{v}_k) = \cosh K_1^*(1+\mathbf{v}_k \tanh K_1^*)$$

and writing $\mathbf{u}_k = \sigma_k^z$ and $\mathbf{v}_k = \sigma_k^x$, it is easy to identify $\mathbf{t}_n(\lambda)$ with the standard Ising transfer matrix:

$$\mathbf{t}_{\mathrm{Ising}} = \exp\left(\sum_{k=1}^{n} K_1^*\sigma_k^x\right)\exp\left(\sum_{k=1}^{n} K_1\sigma_{k-1}^z\sigma_k^z\right), \qquad \tanh K_1^* = \frac{b}{a}, \qquad \tanh K_1 = \frac{bc}{\varkappa}.$$

7. Conclusion

This paper is devoted to the solution of the eigenvalue and eigenvector problems for the finite-size inhomogeneous periodic Baxter–Bazhanov–Stroganov quantum chain model. We use an approach which had been developed in full detail for the quantum Toda chain in [9, 10] and in [11] for the relativistic deformation of the Toda chain. This approach consists of two main steps. In order to find eigenvectors for the transfer matrix $A_n(\lambda) + D_n(\lambda)$, first we find the eigenvectors of the off-diagonal operator $B_n(\lambda)$ adapting the well-known recurrent procedure described in [10] to our root-of-unity case. Then, using these eigenvectors, we construct the eigenvectors for the BBS transfer matrix and show that the coefficients of the decompositions of one set of eigenvectors in terms of the other set factorize into a product of single variable functions, each satisfying the Baxter-type equation. We show that the condition for these equations to have a nontrivial solution is equivalent to the functional relations for the transfer matrix eigenvalues in the BBS or $\tau^{(2)}$ model. In the case of $N = 2$, the Baxter equation can be solved and as a result we obtain the eigenstates of the transfer matrix of the generalized Ising model at the free fermion point. We briefly give the relation of the $N = 2$ BBS model parameters to the standard Ising model parametrization.

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Appendix. Amplitudes $r_{m,k}$ in the homogeneous case

The determination of the amplitudes $r_{m,k}$ for the inhomogeneous BBS model had been reduced to solving equation (47) with (51). Here we show that in the *homogeneous* case this task simplifies to solving just one quadratic equation, using a trigonometric parametrization.

In the homogeneous case, we have

$$\begin{aligned} &a_m = a, \qquad b_m = b, \qquad c_m = c, \qquad d_m = d, \\ &\varkappa_m = \varkappa, \qquad r_m = r, \qquad \mathcal{L}_m(\lambda^N) = \mathcal{L}(\lambda^N) \end{aligned} \tag{A.1}$$

and

$$\begin{pmatrix} \mathcal{A}_m(\lambda^N) & \mathcal{B}_m(\lambda^N) \\ \mathcal{C}_m(\lambda^N) & \mathcal{D}_m(\lambda^N) \end{pmatrix} = (\mathcal{L}(\lambda^N))^m. \tag{A.2}$$

Using the fact that a 2×2 matrix $\mathbf{M}$ with eigenvalues μ_+ and μ_- satisfies

$$\mathbf{M}^m = \frac{\mu_+^m - \mu_-^m}{\mu_+ - \mu_-}\mathbf{M} - \frac{\mu_+^m \mu_- - \mu_-^m \mu_+}{\mu_+ - \mu_-}\mathbf{1},$$

from the matrix $\mathcal{L}(\lambda^N)$ we obtain

$$\mathcal{B}_m(\lambda^N) = -\epsilon\lambda^N r^N \frac{x_+^m - x_-^m}{x_+ - x_-},$$

where $x_+(\lambda^N)$ and $x_-(\lambda^N)$ are the eigenvalues of $\mathcal{L}(\lambda^N)$. These eigenvalues are the roots of the characteristic polynomial $x^2 - \tau(\lambda^N)x + \delta(\lambda^N) = 0$:

$$x_\pm = \tfrac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\delta}), \tag{A.3}$$

where, see (50),

$$\tau(\lambda^N) = \operatorname{tr}\mathcal{L}(\lambda^N) = 1 + \frac{b^N d^N}{\varkappa^N} - \epsilon\lambda^N(\varkappa^N + a^N c^N), \tag{A.4}$$

$$\delta(\lambda^N) = \det\mathcal{L}(\lambda^N) = (b^N/\varkappa^N - \epsilon\lambda^N a^N)(d^N - \epsilon\lambda^N c^N \varkappa^N). \tag{A.5}$$

Introducing the variable ϕ by $x_+/x_- = \mathrm{e}^{\mathrm{i}\phi}$, we find that roots of $\mathcal{B}_m$ correspond to roots $\phi_{m,s}$ of $\mathrm{e}^{\mathrm{i}m\phi} = 1$ (without $\phi = 0$):

$$\phi_{m,s} = 2\pi s/m, \qquad s = 1, 2, \ldots, m-1. \tag{A.6}$$

Now we need to find the explicit relation between λ^N and ϕ. We have

$$\tau + \sqrt{\tau^2 - 4\delta} = \mathrm{e}^{\mathrm{i}\phi}(\tau - \sqrt{\tau^2 - 4\delta}) \qquad \text{or} \qquad \tau^2 = 4\delta\cos^2\frac{\phi}{2}. \tag{A.7}$$

Taking into account (A.4) and (A.5), we consider (A.7) as a quadratic equation for λ^N:

$$\begin{aligned} &\lambda^{2N}(a^{2N}c^{2N} + \varkappa^{2N} - 2a^N c^N \varkappa^N \cos\phi) + (b^{2N}d^{2N} + \varkappa^{2N} - 2b^N d^N \varkappa^N \cos\phi)/\varkappa^{2N} \\ &\quad - 2\epsilon\lambda^N\Big((a^N - b^N)(c^N - d^N) + \frac{a^N b^N c^N d^N}{\varkappa^N} \\ &\quad + \varkappa^N - (a^N d^N + b^N c^N)\cos\phi\Big) = 0. \end{aligned} \tag{A.8}$$

The solution $\lambda^N(\phi)$ of this equation describes the relation between the variables λ^N and ϕ. Therefore, we can translate the zeros (A.6) of $\mathcal{B}_m(\lambda^N(\phi))$ in terms of variable ϕ to zeros $\lambda^N(\phi_{m,s})$. From (47) we get

$$r_{m,s}^N = \epsilon\lambda^N(\phi_{m,s}), \qquad s = 1, 2, \ldots, m-1. \tag{A.9}$$

The value of $r_{m,0}^N$ can be found recursively from (60) using (A.1):

$$r_{m,0}^N = r_{m-1,0}^N a^N c^N + r^N \varkappa^{N(m-1)}, \qquad r_{1,0}^N = r^N = a^N - b^N. \tag{A.10}$$

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